Quantum fluctuations of k-strings: A case study

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K strings in Yang-Mills theory can be considered as bound states of k elementary confining strings carrying one unit of colour flux. Current estimates of k-string tension σ_k are very sensitive to the leading corrections due to quantum fluctuations of the string. In this study we address this problem by comparing Polyakov-Polyakov correlators in the fundamental representation ($k = 1$) with the corresponding ones with $k = 2$ in the confining phase of a $\mathbb{Z}_4$ gauge theory in three dimensions. Highly efficient simulation techniques are available in this case. Although the $k = 1$ Polyakov-Polyakov correlator matches nicely with the expected bosonic string effects up to the next-to-leading order, the $k = 2$ Polyakov correlators show large deviations. This is an important source of potential systematic errors in the current estimates of σ_k.

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THE STRING TENSION

The main hypothesis of the effective string description of confinement is that two color sources in a confining gauge theory are bound together by a thin flux tube which can fluctuate like a massless string

This is suggested by the fact that the static quark potential $V(R)$ shows, in first approximation, a linear behaviour at large separation of quarks:

$$V(R) \sim \sigma R,$$

where σ is the string tension

The estimate of string tension σ is very sensitive to systematic errors because it depends of the functional relation used to fit the data

The string tension value can be extracted by looking at the correlations of Polyakov loops in the confined phase:

$$\langle P(x) P^{\dagger}(x+R) \rangle = \exp(-F(R,L)),$$

where $F(R,L)$ is the free energy that depends on the inverse temperature $L = 1/T$ (the lattice size in the time direction) and the distance R

THE STRING TENSION

The free energy $F(R, L)$ is given by a classical and a quantum contribution:

$$F(R, L) = F_{\rm cl}(R, L) + F_{\rm q}(R, L)$$

The classical term corresponds to the area law:

$$F_{\rm cl}(R, L) = \sigma_0 LR + k(L)$$

In the free string model, the quantum term encodes the flux tube fluctuations, and is equal to the free energy of the $d-2$ massless scalar fields describing them:

$$F_{\rm q}(R, L) = (d-2) \log \eta(\tau) \qquad \tau \equiv \frac{iL}{2R}$$

where η is the Dedekind eta function:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} (1 - q^n) \quad ; \quad q = e^{2\pi i \tau}$$

The free string picture is an effective description valid in the infrared limit therefore we expect it is accurate at long distance regime and low temperature

THE STRING TENSION

When the temperature T is closer to deconfinement, the free string contribution is not sufficient to account for the quantum fluctuations of the flux tube and string self-interaction effects become non negligible

The next-to-leading order is also universal [1,2]

It was first calculated in the Nambu-Goto model [3]

Including this contribution $F_{\rm q}(R,L)$ becomes

$$F_q^{(NLO)}(R,L) = (d-2)\left[\log\eta(\tau) - \frac{\pi^2 L}{1152\,\sigma R^3}\left[2E_4(\tau) - E_2^2(\tau)\right]\right] + O\left(\frac{1}{(\sigma L R)^2}\right)$$

where E_2 and E_4 are the Eisenstein functions:

$$\begin{aligned} E_2(\tau) &= 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n \\ E_4(\tau) &= 1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n \\ q &\equiv e^{2\pi i\tau} \end{aligned}$$

where $\sigma(n)$ and $\sigma_3(n)$ are, respectively, the sum of all divisors of n (including 1 and n) and the sum of their cubes

The effectiveness of all previous relations are proved in several papers [4,5,6]

K-STRINGS

In addition to charges in the fundamental representation one can consider the potential between static charges in higher representations of the gauge group $SU(N)$

DEFINITION:
The N-ality of a representation $\mathcal{R}$ (and we indicate it with $k(\mathcal{R})$) is the number of fundamental minus the number of antifundamental representations from which the representation $\mathcal{R}$ is built by tensor products, modulo N

Example: in $SU(3)$, $3 \otimes 3 = 6 \oplus \bar{3}$, so the two representations 6 and $\bar{3}$ have 3-ality $k = 2$

Sources with N-ality $k \neq 0$ are confined

The stable string describing the IR behavior of the associated flux tube is called k-string

Its tension σ_k depends only on the N-ality k

There are two main proposals for the k-string tension ratios σ_k/σ (where $\sigma \equiv \sigma_1$):

- Casimir formula: $\frac{\sigma_k}{\sigma} = \frac{k(N-k)}{N-1}$
- Sine formula: $\frac{\sigma_k}{\sigma} = \frac{\sin(k\pi/N)}{\sin(\pi/N)}$

In the effective string picture, we can interpret a k-string as a bound state of k elementary fundamental strings

A QUESTION

QUESTION: Can we use the functional relations obtained for the fundamental string tension in the general case of k-strings ?

We think that the string energy appears not only as mechanical energy of the fluctuating string but also as excitation energy of the internal states of the k-string

If this is the case, there are serious doubts about the possibility that the well-acquainted functional relations valid for the fundamental string can be applied also to the k-string

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In order to test this idea it is convenient to perform the numerical simulations in the simplest model where there are two different k-strings: the $3d$ $\mathbb{Z}_4$ gauge model

It is known that this model can be translated into a spin model by the so called Kramers-Wannier duality transformation as much as any physical property of the gauge system can be translated into a corresponding property of its spin dual

In this case the dual is a spin model with $\mathbb{Z}_4$ global symmetry which can be written as a symmetric Ashkin-Teller (AT) model

ASHKIN-TELLER MODEL

The AT model is formed by two coupled, ferromagnetic, Ising systems defined by the two-parameter action:

$$S = -\sum_{\langle xy \rangle} \beta \left(\sigma_x \sigma_y + \tau_x \tau_y\right) + \alpha \; \sigma_x \sigma_y \tau_x \tau_y ,$$

where σ_x and τ_x are the Ising variables $(\sigma_x, \tau_x = \pm 1)$

The global $\mathbb{Z}_4$ symmetry of the action is generated by the transformation $\sigma \to -\tau, \quad \tau \to \sigma$

An independent $\mathbb{Z}_2$ symmetry is generated by the transformation $\sigma \leftrightarrow \tau$, which is related to the charge conjugation of the corresponding dual model

In particular, it is possible to show that the AT model is dually equivalent to two coupled $\mathbb{Z}_2$ gauge models; actually, the partition function is:

$$Z_{AT}(\alpha, \beta) \propto \sum_{U_l = \pm 1, V_l = \pm 1} \prod_P e^{b(U_P + V_P) + a U_P V_P}$$

The great advantage to study the $3d$ AT model instead of the original $3d$ $\mathbb{Z}_4$ gauge model is that a non local cluster updating algorithm can be used and therefore very high precisions can be reached

ALGORITHM

We use a very efficient cluster algorithm introduced by Swensen and Wang

It is possible to show that the expectation value of a Wilson loop of the gauge model $\langle W(C)\rangle_{gauge}$ is given by:

$$\langle W(C)\rangle_{gauge} = \frac{Z^*_{AT}}{Z_{AT}},$$

where Z_{AT} is the partition function of the AT model and Z^*_{AT} is that modified by a suitable twist of the couplings

In order to determine the $\langle W(C)\rangle_{gauge}$ in the fundamental representation (k=1) it suffices to twist the variable $\sigma_x \to -\sigma_x$ (or $\tau_x \to -\tau_x$)

Similarly, flipping the signs of both spins σ_x and τ_x we get the Wilson loop in the k=2 representation

Actually, because we have a model written in terms of Ising variables we can use a very powerful method to estimate Wilson loops based on the Fortuin-Kasteleyn (FK) clusters

For each FK configuration one looks for paths in the cluster:

- $W(C) = 1$ if there is no path linked with the loop C
- $W(C) = 0$ otherwise

The same ideas does apply to determine the Polyakov-Polyakov correlator $\langle P(0)P^{\dagger}(R)\rangle$

SIMULATIONS

We have simulated the AT model in two different points in the space parameters (α, β):

- C1=(0.05,0.2070)
- C2=(0.07,0.1975) [very preliminary results]

where the string tensions are determined to be:

- C1 $\rightarrow \sigma_1 = 0.02084(5)$, $\sigma_2 = 0.0323(5)$
- C2 $\rightarrow \sigma_1 = 0.01560(1)$, $\sigma_2 = 0.0210(6)$

and where the tension ratio is:

- C1 $\rightarrow \sigma_2/\sigma_1 \simeq 1.55$
- C2 $\rightarrow \sigma_2/\sigma_1 \simeq 1.35$

Note the values obtained by the Sine and Casimir formula in our case (k=2, N=4):

- Casimir $\rightarrow 4/3 \simeq 1.33$
- Sine $\rightarrow \sqrt{2} \simeq 1.41$

We have taken 10^6 measurements for both σ_1 and σ_2 on a lattice $64^2 \times L$

The value of L has been determined in such a that $T \simeq T_c/2$ using the relation $T_c/\sqrt{\sigma} \simeq 1.1$

In the following:
$\sigma_1 \equiv \sigma_f$ and $\sigma_2 \equiv \sigma_{ff}$

ANALYSIS

In order to decide if the functional relation between σ_k and R is correct two conditions are needed:

- The fitting parameters (in particular σ_k) should be stable at large R
- The estimated value of σ_k should not depend on T since the NLO accounts the T dependence

RESULTS

In Fig. 1 it is possible to see the σ_f value interpolated by a leading order (LO) functional relation in a range $[R_{min}, 18]$ (where R_{min} is the value that appears on the x-axes); albeit for big values of R_{min} it seems to appear a plateau, there are three different values of σ_f, therefore the functional relation (FR) used is not correct to determine it

Fig. 2 is similar to Fig. 1, but in this case we have used the same functional relation to determine the value of σ_2; also in this case the FR is not good to determine it

In Fig. 3 we interpolated the value of σ_f with two different FRs, one obtained by a LO approximation and the other one by NLO; the value of σ_f increases significantly and the plateau appears for a smaller value of R_{min}

RESULTS

In Fig. 4 we plot the values of $\chi^2/d.o.f.$ for interpolations appear in Fig. 3; note that for $R_{min} \geq 11$ there is no difference in $\chi^2/d.o.f.$ between LO and NLO, despite the values of σ_f in the same range (Fig. 3) are different;

it is interesting to note that in NLO case also the value of $\chi^2/d.o.f.$ shows a plateau since $R_{min} \geq 9$, therefore this FR is better to describe what happens at small values of R

Fig. 5 is similar to Fig. 1 but now we use FRs obtained with a NLO approximation. Now the FR is actually correct: the value of σ is stable also for small value of R_{min} and there is only one value of it for three different values of L (different values of T)

In Fig. 6 and Fig. 7 we interpolate the value of σ_2 with NLO approximation with two different hypotheses: in Fig. 6 the two strings are stuck together and fluctuate as a single string; in Fig. 7 the two strings are independent. It is clear that in this case the FR is not correct to determine accurately σ_2: in both cases a plateau does not appear for σ_2 and there are different values for different values of T.

As a consequence σ_2 is affected by systematic errors

σ_f vs R_{min} for three different T at LO

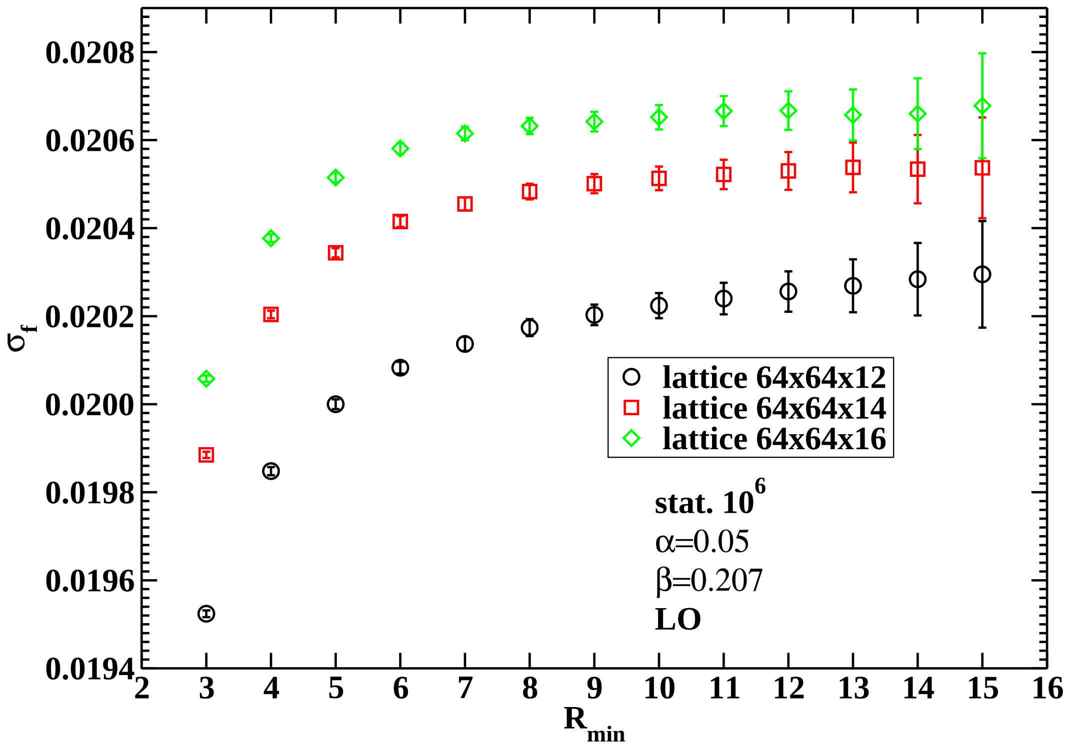

FIGURE 1

σ_{ff} vs R_{min} for three different T at LO

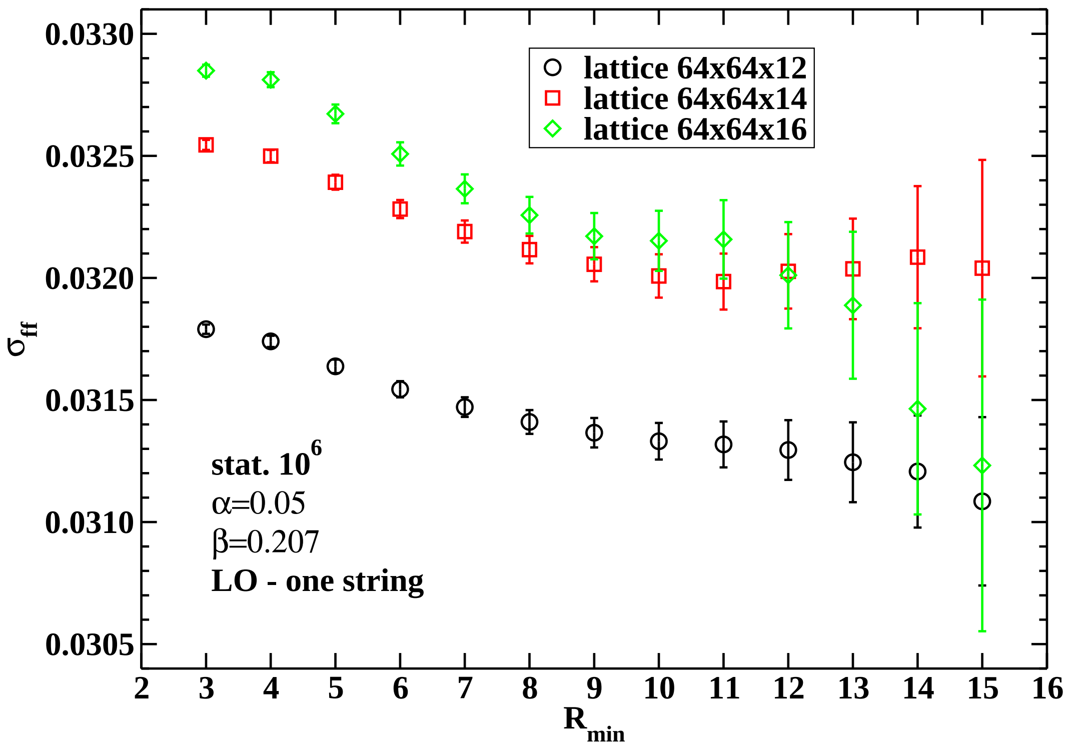

FIGURE 2

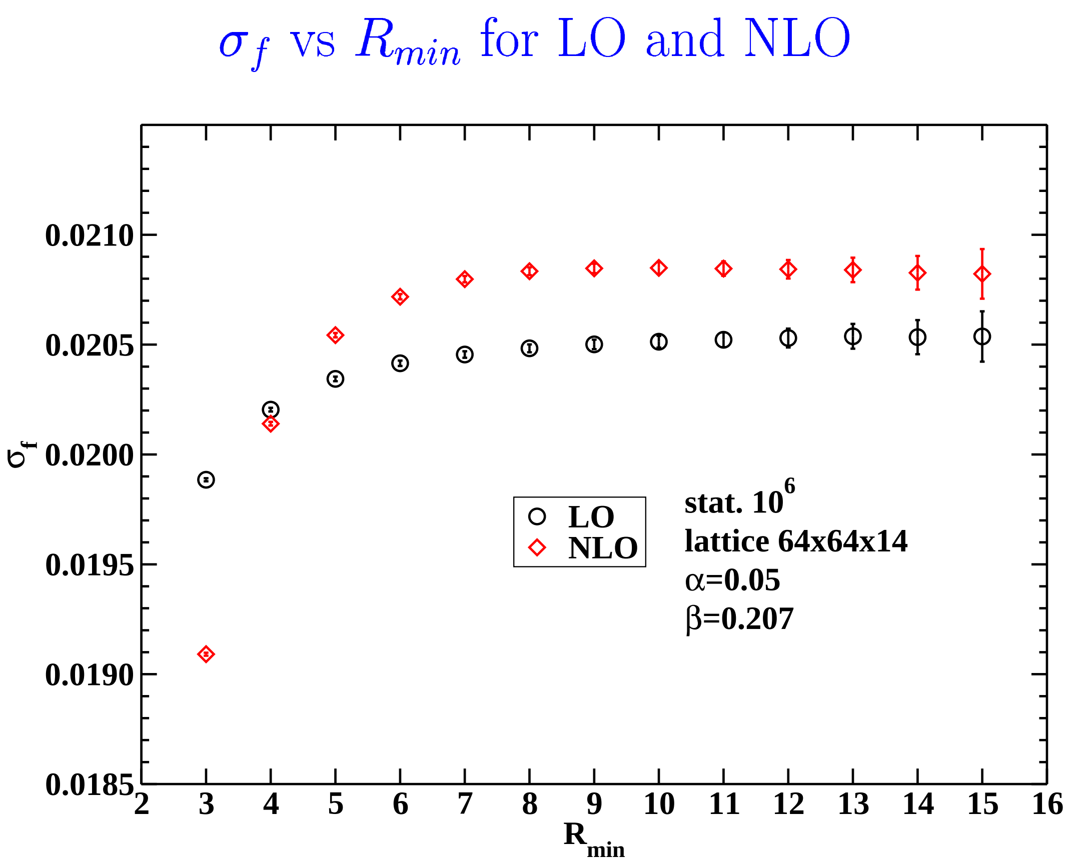

FIGURE 3

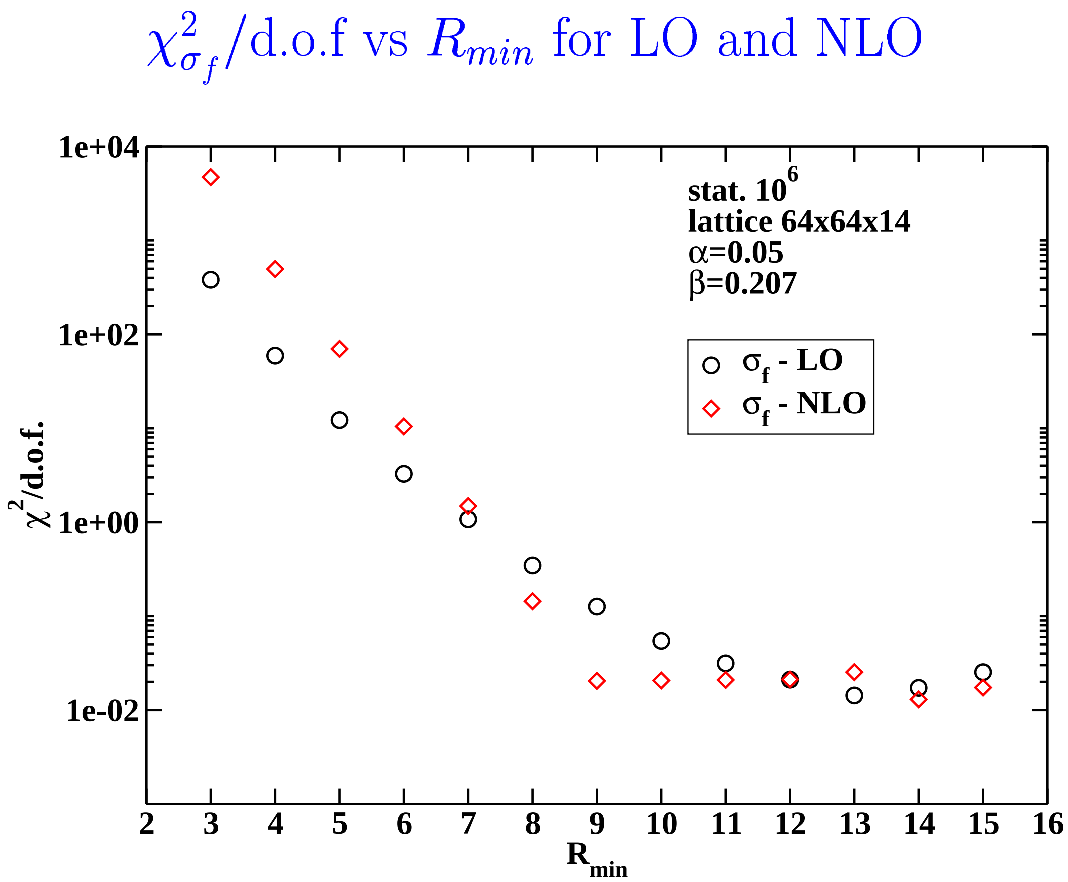

FIGURE 4

σ_f vs R_{min} for three different T at NLO

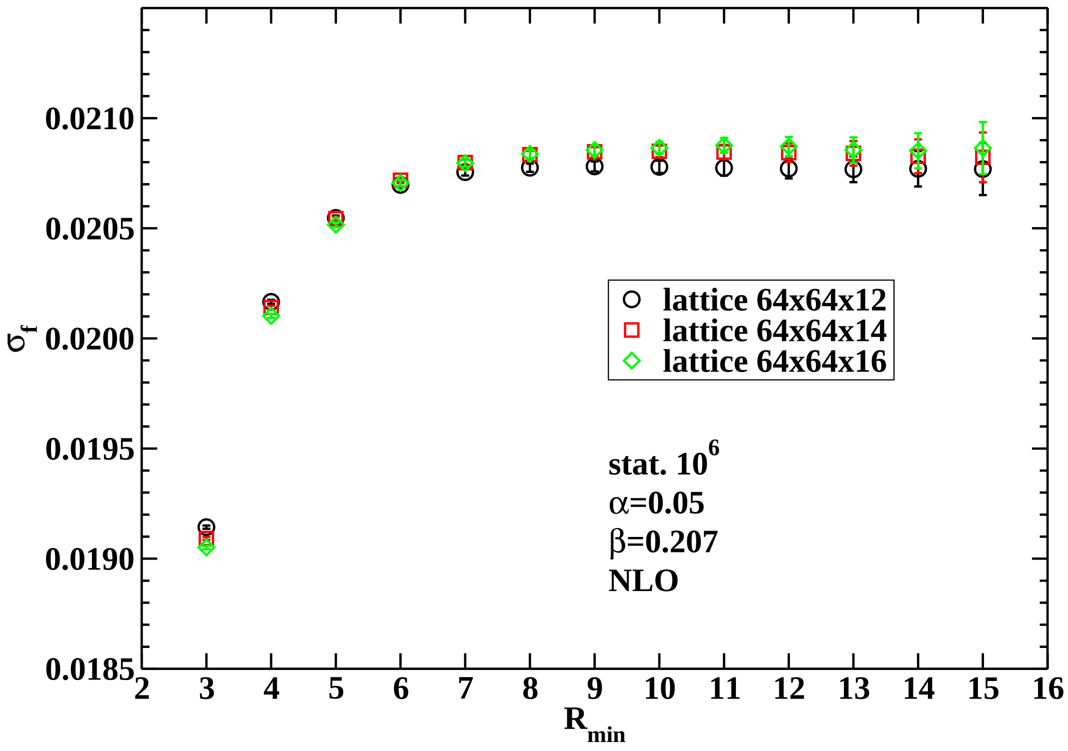

FIGURE 5

σ_{ff} vs R_{min} for three different T at NLO (one string)

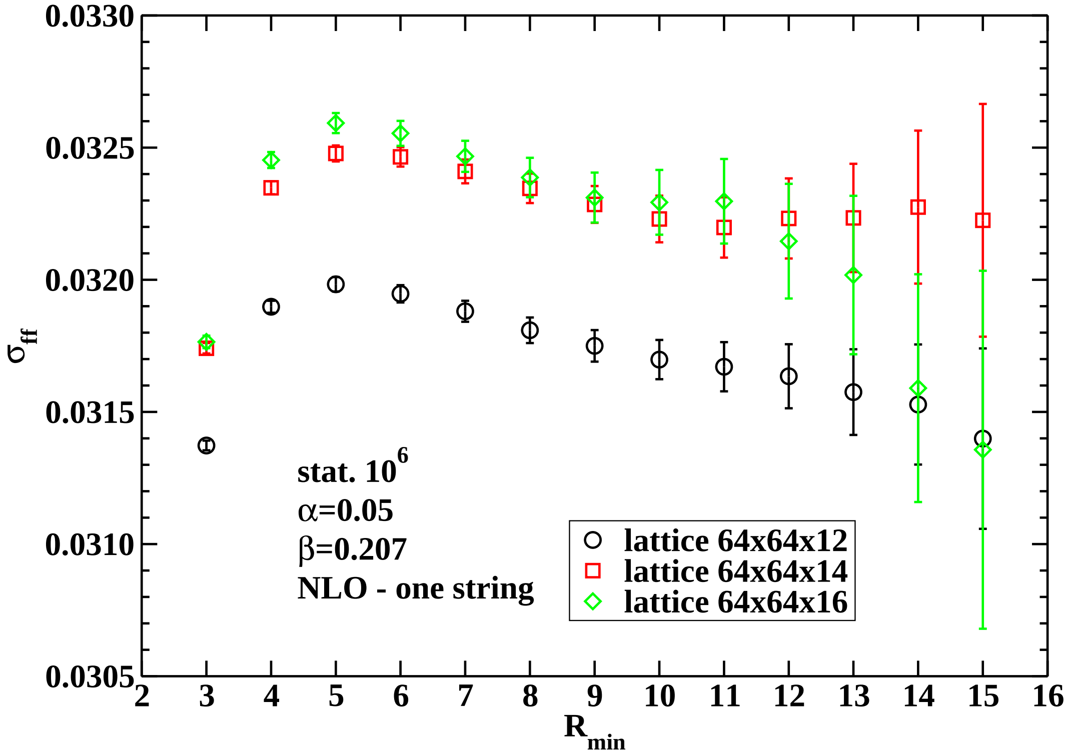

FIGURE 6

σ_{ff} vs R_{min} for three different T at NLO (two strings)

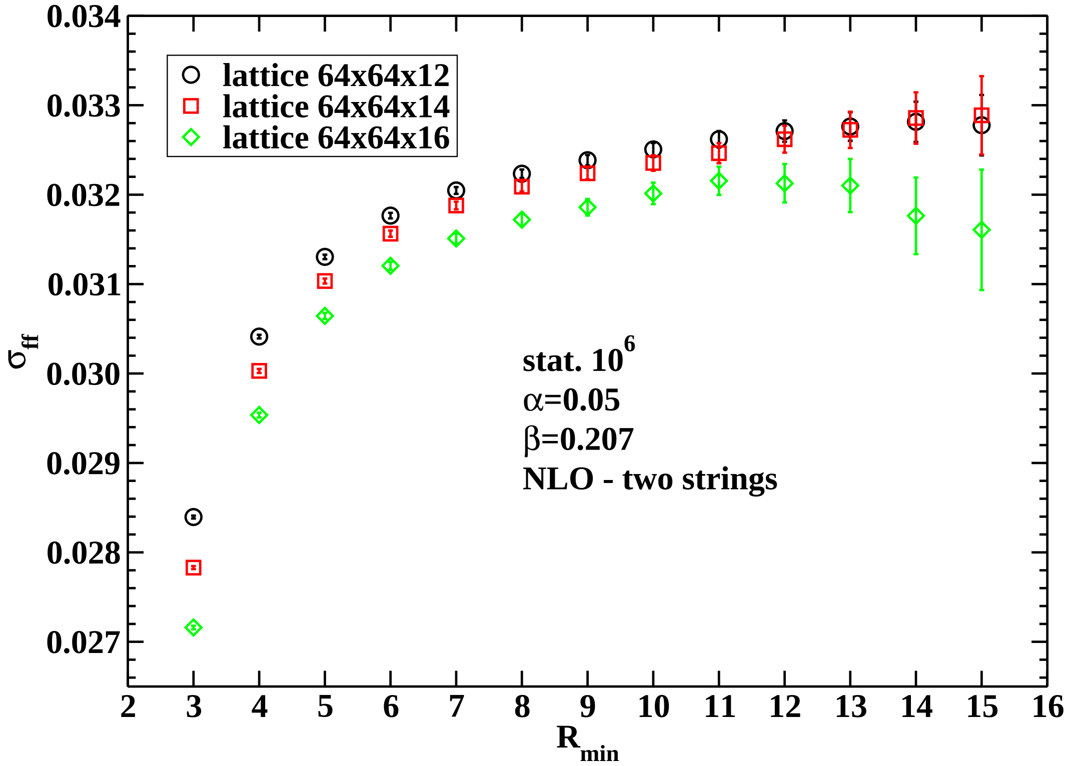

FIGURE 7

CONCLUSIONS

The functional relations which are extremely suitable to fit data related to fundamental string tension (k=1) are not adequate to describe 2-string tension

An accurate estimate of the k-string tension is rather problematic:

- from a numerical point of view a blind application of the usual formulas for the Polyakov correlators introduces strong systematic errors
- from a theoretical point of view it is necessary to face the problem to determine the correct functional relation of Polyakov-Polyakov correlator for higher representations, which is presently unknown

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