Operator product expansion on the lattice: analytic Wilson coefficients

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We present first results for Wilson coefficients up to order spin $n = 4$ for the case of Wilson fermions. They are derived from the off-shell Compton scattering amplitude $W_{\mu\nu}(a, p, q)$ of massless quarks with momentum p. The Wilson coefficients are classified according to their transformation under the hypercubic group H(4). We give selected examples for a special choice of the momentum transfer q.

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Introduction

Computation of moments of structure functions is still a challenge in particle physics

Theoretically it is based on the relation between moment sum rules and the operator product expansion (OPE) of the Compton scattering amplitude of a photon ($\hat{J}_\mu(q)$) off a nucleon target ($\big|N(p)\rangle$):

$$
\begin{aligned}
W_{\mu\nu} &= \langle N(p)\big|\hat{J}_\mu(q)\hat{J}^\dagger_\nu(q)\big|N(p)\rangle \qquad (1)\\
&= \sum_{m,n} C^m_{\mu\nu,\mu_1,\dots\mu_n}(q/\mu)\langle N(p)\big|\mathcal{O}^m_{\mu_1,\dots\mu_n}\big|N(p)\rangle
\end{aligned}
$$

The Wilson coefficients $C^m_{\mu\nu,\mu_1,\dots\mu_n}(q/\mu)$ are universal and usually calculated in perturbation theory. In this framework, however, OPE is ambiguous due to renormalon problems.

Putting the OPE on the lattice the problem becomes well defined.

This requires both matrix elements and Wilson coefficients to be computed on the lattice.

Defined on the lattice the sum in (1) runs over all possible operators $\mathcal{O}^m_{\mu_1,\dots\mu_n}$ of spin n. m distinguishes operators of same spin transforming under different irreducible representations of the hypercubic symmetry.

Introduction

For the OPE to converge we need small p : $p^2 \ll q^2$ and $p^2 \ll (2\pi/a)^2$

In order to match continuum for $W_{\mu\nu}$ we have to demand: $p, q \ll 1/a$.

This results in the condition: $p^2 \ll q^2 \ll (2\pi/a)^2$

$\longrightarrow$ It is crucial to investigate possible lattice artefacts and a safe range for aq for the Wilson coefficients

On the lattice both the non-perturbative Wilson coefficient $C^{m}_{\mu\nu,\mu_1\ldots\mu_n}(a,q)$ and the Wilson coefficient at Born level $C^{BORN,m}_{\mu\nu,\mu_1\ldots\mu_n}(a,q)$ have $\mathcal{O}(a^2q^2)$ corrections:

$$
\begin{aligned}
C^{BORN,m}_{\mu\nu,\mu_1\ldots\mu_n}(a,q) &= C^{(0)BORN,m}_{\mu\nu,\mu_1\ldots\mu_n} + (a^2q^2)\, C^{(2)BORN,m}_{\mu\nu,\mu_1\ldots\mu_n} + \ldots \\
C^{m}_{\mu\nu,\mu_1\ldots\mu_n}(a,q) &= c_n^{(0)m}(q^2) C^{(0)BORN,m}_{\mu\nu,\mu_1\ldots\mu_n} + \\
& \quad c_n^{(2)m}(q^2)(a^2q^2)\, C^{(2)BORN,m}_{\mu\nu,\mu_1\ldots\mu_n} + \ldots
\end{aligned}
$$

The quantities $c_n^{(i)m}(q^2)$ are the so-called reduced Wilson coefficients.

Introduction

The $\mathcal{O}(a^2q^2)$ correction is given by the ratio

$$\frac{C^{m}_{\mu\nu,\mu_1\ldots\mu_n}(a,q)}{C^{BORN,m}_{\mu\nu,\mu_1\ldots\mu_n}(a,q)} = c_n^{(0)m}(q^2) + \left(c_n^{(2)m}(q^2) - c_n^{(0)m}(q^2)\right)\frac{C^{(2)BORN,m}_{\mu\nu,\mu_1\ldots\mu_n}}{C^{(0)BORN,m}_{\mu\nu,\mu_1\ldots\mu_n}}(a^2q^2) + \ldots$$

$\longrightarrow$ It is essential to determine the Born Wilson coefficient $C^{BORN,m}_{\mu\nu,\mu_1\ldots\mu_n}(a,q)$ in its analytic form in order to extract the $C^{(i)BORN,m}_{\mu\nu,\mu_1\ldots\mu_n}$.

In order to carry out this program we do the following steps:

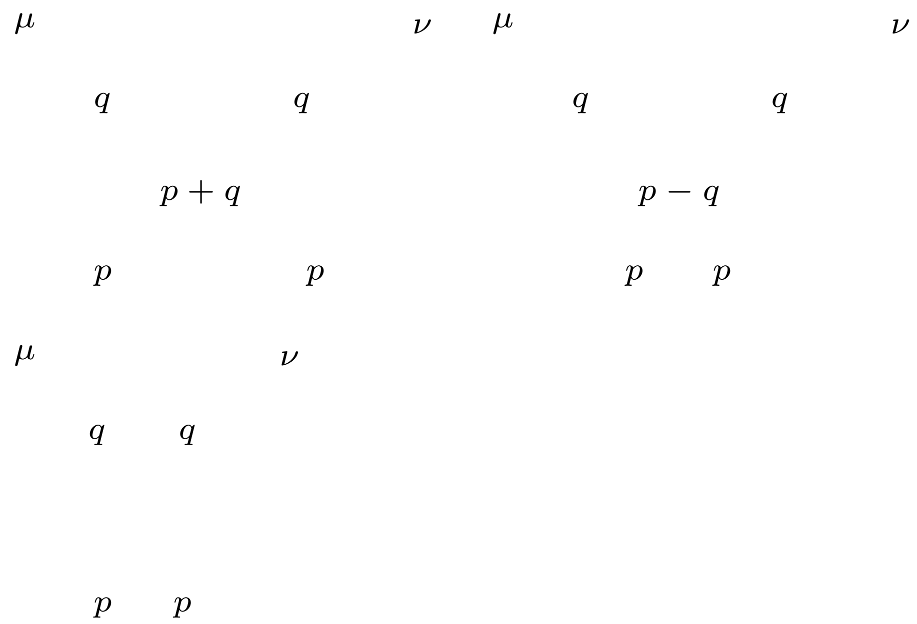
- Expanding the lattice amplitude $W_{\mu\nu}(a,q,p)$ in powers of $\sin^n(ap)$ $(n \leq 3)$ of external quark momentum p
- Division into tree level matrix elements and Wilson coefficients
- Classification under the transformation under hypercubic group H(4)

Currently we use the Wilson fermion representation of quarks. In this respect the results are an extension of a program initiated by D. Petters[1]

[1] D. Petters, Dissertation, 2000

Calculation

The following lattice Feynman diagrams contribute to the Compton scattering amplitude at tree level:



We calculate the amplitude $W_{\mu\nu}(a,p,q)$ for off-shell massless quark states with momentum p.

All manipulations are performed symbolically by a *Mathematica* program package

Calculation

Calculational details:

- $W_{\mu\nu}(a,q,p)$ is computed using symbolic lattice Feynman rules for off-shell Wilson fermions in momentum space
- All combinations of γ matrices are expanded into the general base of 4×4 matrices: $1, \gamma_5, \gamma_\mu, \gamma_\mu\gamma_5, \sigma_{\mu\nu}$
- For Wilson fermions we get the following tower of local operators (up to three covariant derivatives)
  - Unpolarised case:
    $\bar{\psi}1\psi\,, \bar{\psi}\gamma_\mu \overset{\leftrightarrow}{\mathrm{D}}_\nu \psi\,, \bar{\psi} \overset{\leftrightarrow}{\mathrm{D}}_\mu \overset{\leftrightarrow}{\mathrm{D}}_\nu \psi\,, \bar{\psi}\gamma_\mu \overset{\leftrightarrow}{\mathrm{D}}_\nu \overset{\leftrightarrow}{\mathrm{D}}_\omega \overset{\leftrightarrow}{\mathrm{D}}_\rho \psi$
  - Polarised case (from non-diagonal part of $W_{\mu\nu}$):
    $\bar{\psi}\gamma_\mu\gamma_5\psi\,, \bar{\psi}\sigma_{\mu\nu} \overset{\leftrightarrow}{D}_\omega \psi\,, \bar{\psi}\gamma_\mu\gamma_5 \overset{\leftrightarrow}{D}_\nu \overset{\leftrightarrow}{D}_\omega \psi\,, \bar{\psi}\sigma_{\mu\nu} \overset{\leftrightarrow}{D}_\omega \overset{\leftrightarrow}{D}_\rho \overset{\leftrightarrow}{D}_\lambda \psi$
- Expansion of $W_{\mu\nu}(a,q,p)$ in powers of $\sin(ap)$
- Projection onto the tree level matrix elements using the momentum space correspondence $\overset{\leftrightarrow}{D}_\mu \sim \sin(ap_\mu)$
- Decompose the corresponding operators into the base of operators transforming irreducibly under the hypercubic group [2]
- Collection of the coefficients belonging to the respective bases $\longrightarrow$ Wilson coefficients at Born level

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[2] M. Göckeler et al., PR D54 (1996) 5705

Results: $\sim \sin^0(ap)$

The general result for the expansion of $W_{\mu\nu}(a,p,q)$ in powers of $\sin^n(ap)$, $(n \leq 3)$ is too large as to be presentable in closed form (880 terms).

Let us start with the part $\sim \sin^0(ap)$. Here we can present the corresponding part of the amplitude in most general form:

$$
\begin{aligned}
W^{(0)}_{\mu\nu}(a,q)/e_\gamma^2 \quad = \quad & -a\,r\,\delta_{\mu\nu}\,\bar\psi 1\psi - \frac{8\,\mathrm{a\,r}\cos(\mathrm{a}/2\mathrm{q}_\mu)^2}{\mathrm{Q}^2}\,\delta_{\mu\nu}\,\bar\psi 1\psi + \\
& \sum_\tau \frac{2\,a\,r\,\cos(a/2q_\mu)^2\cos(aq_\tau)}{Q^2}\,\delta_{\mu\nu}\,\bar\psi 1\psi + \\
& \frac{8\,a\,r^3\,\sin(a/2q_\mu)\sin(a/2q_\nu)}{Q^2}\,\bar\psi 1\psi - \\
& \sum_\tau \frac{2\,a\,r^3\,\cos(aq_\tau)\sin(a/2q_\mu)\sin(a/2q_\nu)}{Q^2}\,\bar\psi 1\psi + \\
& \frac{2\,a\,r\,\cos(a/2q_\mu)\sin(a/2q_\nu)\sin(aq_\mu)}{Q^2}\,\bar\psi 1\psi + \\
& \frac{2\,a\,r\,\cos(a/2q_\nu)\sin(a/2q_\mu)\sin(aq_\nu)}{Q^2}\,\bar\psi 1\psi + \\
& \sum_{\tau,\sigma} \frac{2\,i\,a\,\cos(a/2q_\mu)\cos(a/2q_\nu)\sin(aq_\sigma)}{Q^2}\,\bar\psi\gamma_\tau\gamma_5\epsilon_{\mu\sigma\nu\tau}\psi\,,
\end{aligned}
$$

with

$$
Q^2(a,q) = \sum_\tau \sin(aq_\tau)^2 + r^2\left(\sum_\tau (1-\cos(aq_\tau))\right)^2
$$

(e_γ denotes the quark-photon-coupling. In the following we will set $a=1, r=1$.)

Results: $\sim \sin^0(ap), \sim \sin^3(ap)$

Maximal symmetric case:
$q = (f, f, f, f), c = \cos(f), s = \sin(f)$

For the diagonal part $W_{11}^{(0)}$ we find the decomposition

$$W_{11}^{(0)}(q)/e_\gamma^2 = -\frac{6(3-c)(1-c)}{Q(c,s)}\bar{\psi}1\psi$$

with

$$Q(c,s) = 4s^2 + 16(1-c)^2$$

The off-diagonal part $W_{12}^{(0)}$ has contributions for polarised structure functions also

$$W_{12}^{(0)}(q)/e_\gamma^2 = \frac{2(3-c)(1-c)}{Q(c,s)}\bar{\psi}1\psi + \frac{\mathrm{i}(1+c)\mathrm{s}}{\mathrm{Q}(\mathrm{c},\mathrm{s})}(\bar{\psi}\gamma_3\gamma_5\psi - \bar{\psi}\gamma_4\gamma_5\psi)$$

At this level there is no need to decompose the operators into H(4) bases - they still form an independent set under the hypercubic group. The Wilson coefficients can simply be read off.

Let us turn to the most complicated case in our expansion: $W_{\mu\nu}^{(3)}(q)$ - the contribution of order $\mathcal{O}(\sin^3(ap))$ to the Compton scattering amplitude

We restrict to the diagonal case $W_{11}^{(3)}(q)$: here we have a operator base of dimension 256. The elements are classified into the 20 inequivalent irreducible representations of H(4) and charge conjugation property ($C = \pm 1$).

Results: $\sim \sin^3(ap)$

We have found only 14 independent combinations $b_i(c,s)$ from which all Wilson coefficients are built: (the results are given in the form $a_i(c,s) = b_i(c,s)\ \cdot Q^4(c,s)$)

$$
\begin{aligned}
a_1 &= 16i(1-c)^2(-4+3c)(-696+2000c-2172c^2+1107c^3-270c^4+27c^5) \\
a_2 &= -16i(1-c)^2(1+c)(-692+2060c-2394c^2+1326c^3-333c^4+27c^5) \\
a_3 &= -16i(1-c)^2(388-1280c+1736c^2-1240c^3+493c^4-102c^5+9c^6) \\
a_4 &= 16i(1-c)^2(1+c)(-4+3c)(4-9c+3c^2)(6-8c+3c^2) \\
a_5 &= -16i(1-c)^2(1+c)(68-116c+28c^2+52c^3-39c^4+9c^5) \\
a_6 &= -32i(1-c)^2(1+c)(178-382c+238c^2-2c^3-39c^4+9c^5) \\
a_7 &= 96i(1-c)^2(1+c)^2(14-64c+88c^2-48c^3+9c^4) \\
a_8 &= 16i(1-c)^2(1+c)(4-3c)^2(4-9c+3c^2) \\
a_9 &= -16i(1-c)^2(-112+290c-296c^2+147c^3-42c^4+9c^5) \\
a_{10} &= -16i(1-c)^2(1+c)(-112+242c-162c^2+21c^3+9c^4) \\
a_{11} &= 16i(1-c)^2(1+c)(-88+128c-28c^2-27c^3+9c^4) \\
a_{12} &= -16i(1-c)^2(-4+3c)(-34-12c+87c^2-54c^3+9c^4) \\
a_{13} &= 32i(1-c)^2(1+c)(72-188c+184c^2-75c^3+9c^4) \\
a_{14} &= -96i(1-c)^2(1+c)^2(-4+3c)(6-8c+3c^2)
\end{aligned}
$$

Results: $\sim \sin^3(ap)$, Summary

The following table presents some examples for Wilson coefficients belonging to definite irreducible representations and C-parity

representation	C-parity	Wilson coefficient
$\tau_1^{(1)}$	$+1$	$\frac{1}{2\sqrt{2}}(b_1 + 3b_3 + 3b_9 + 3b_{12})$
$\tau_1^{(1)}$	-1	$-\frac{3}{2\sqrt{6}}(b_3 + 3b_9)$
$\tau_1^{(3)}$	$+1$	$\frac{\sqrt{3}}{4}(b_1 + b_3/3 - b_9/3 - b_{12})$
$\tau_3^{(6)}$	$+1$	$\frac{\sqrt{3}}{4\sqrt{2}}(b_2 + b_4 + b_8 + b_{11}) + \frac{1}{2\sqrt{6}}(b_{10} + b_{13})$

The normalisation of the Wilson coefficients is determined up to overall factors due to the operator base normalisation

Conclusion and summary

- We present first results for analytic Wilson coefficients on the lattice for Wilson fermions
- We restrict the order of the corresponding operators up tp three covariant derivatives
- The Wilson coefficients are given as functions of $\sin(aq)$ and $\cos(aq)$ which allows the determination of $\mathcal{O}(a^2q^2)$ dependence
- The calcluation will be extended to overlap fermions which have a smaller set of possible operators