Strong to weak coupling transitions of $SU(N)$ gauge theories in $2+1$ dimensions

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25th July 2006

F.B., Mike Teper: hep-th/0511081

Contents

Introduction

There are several known strong to weak coupling transitions in $SU(N)$ gauge theories:

- The Gross–Witten transition at $N = \infty$ in $D = 1 + 1$.
- A non–analyticity in Wilson loop eigenvalue spectra at $N = \infty$ in the $D = 1 + 1$ continuum theory.
- The bulk transition for $N \geq 5$ in $D = 3 + 1$.
- There is a rapid crossover observed from perturbative to non–perturbative physics in QCD.

Do these have anything in common? What about $D = 2 + 1$?

Gross–Witten transition

$D = 1 + 1$ $SU(N)$ lattice gauge theory can be solved exactly. For $N = \infty$ it is evaluated by considering the eigenvalue density $\rho(\alpha)$.

- For strong coupling the trace of the plaquette is small, so the eigenvalue density is nearly uniform.
- For weak coupling the trace $\to 1$, so the distribution becomes peaked around $\alpha = 0$

In between there is a third–order phase transition at $\gamma \equiv \frac{\beta}{2N^2} = 0.5$. This is a strong to weak coupling transition. At the transition a gap opens in the density of eigenvalues — on the weak coupling side the density is zero for $|\alpha| > \alpha_c$.

[*D. Gross, E. Witten, Phys. Rev. D21 (1980) 446*]

Wilson loops

The Gross–Witten transition happens at a critical value of the bare coupling, i.e. when the plaquette passes through a critical length scale.

What happens to other Wilson loops when they pass through this length scale?

In fact there is a similar transition in the continuum theory. A gap opens at a critical area

$$A_c = \frac{8}{g^2 N} \tag{1}$$

[*B. Durhuus and P. Olesen, Nucl. Phys. B184 (1981) 461*]

[*A. Bassetto, L. Griguolo and F. Vian, Nucl. Phys. B559 (1999) 563*]

However, when this non–analyticity occurs,

- The partition function is analytic
- The trace of the Wilson loop and all its powers is analytic

The physical significance of this transition is unclear.

The bulk transition

In $D = 3 + 1$, there is a strong first–order phase transition for $N \geq 5$. This is also a strong to weak coupling transition. For $N = \infty$ a gap opens in the plaquette eigenvalue spectrum.

Is there a transition for Wilson loops, like the one in $D = 1 + 1$? If so this could explain the observed rapid transition from perturbative to non–perturbative physics.

[*R. Narayanan and H. Neuberger, hep-lat/0501031, hep-lat/0509014*]

Answering this requires looking at the tails of the eigenvalue distribution, and extrapolating to $N = \infty$.

So we begin by looking in $D = 2 + 1$.

Phase transitions

At a phase transition, derivatives of the partition function diverge or are discontinuous.

First derivative is the average plaquette $\langle u_p \rangle$.

Second derivative is the specific heat:

$$C = N_p(\langle \overline{u_p}^2 \rangle - \langle \overline{u_p} \rangle^2). \tag{2}$$

Third derivative is:

$$C' = N_p^2(\langle \overline{u_p}^3 \rangle - 3\langle \overline{u_p} \rangle \langle \overline{u_p}^2 \rangle + 2\langle \overline{u_p} \rangle^3). \tag{3}$$

We define $C_2 = N^2 \times C$ and $C_3 = N^4 \times C'$.

At $N = \infty$, there are an infinite number of degrees of freedom at each point. So phase transitions can occur on a finite volume.

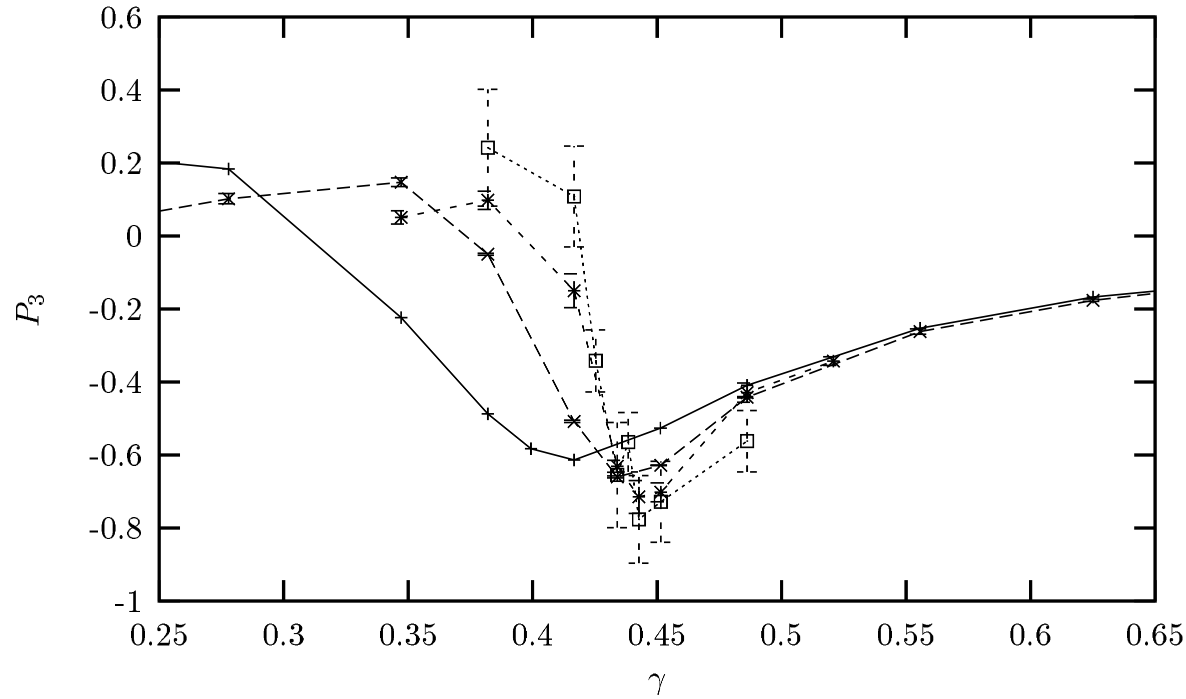
So we also calculate local versions of C_2 and C_3:

$$P_2 = N^2 \times (\langle {u_p}^2 \rangle - \langle u_p \rangle^2) \tag{4}$$

$$P_3 = N^4 \times (\langle {u_p}^3 \rangle - 3\langle u_p \rangle \langle {u_p}^2 \rangle + 2\langle u_p \rangle^3). \tag{5}$$

Results

We see a very clear discontinuity developing in P_3 as $N \to \infty$:

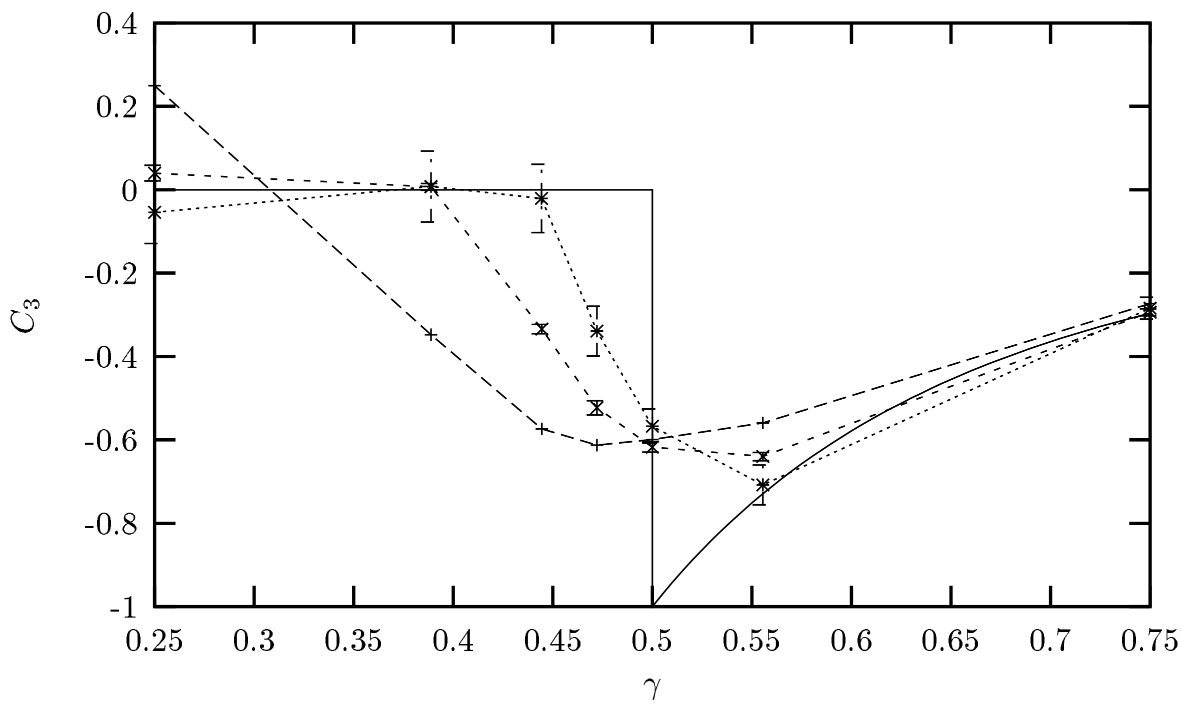


P_3, as a function of $\gamma = \frac{\beta}{2N^2}$ for SU(6) (+), SU(12) ($\times$), SU(24) ($*$) and SU(48) ($\Box$).

So there is a 'local' third order transition. Only at $N = \infty$, not for finite N.

Comparison to Gross–Witten

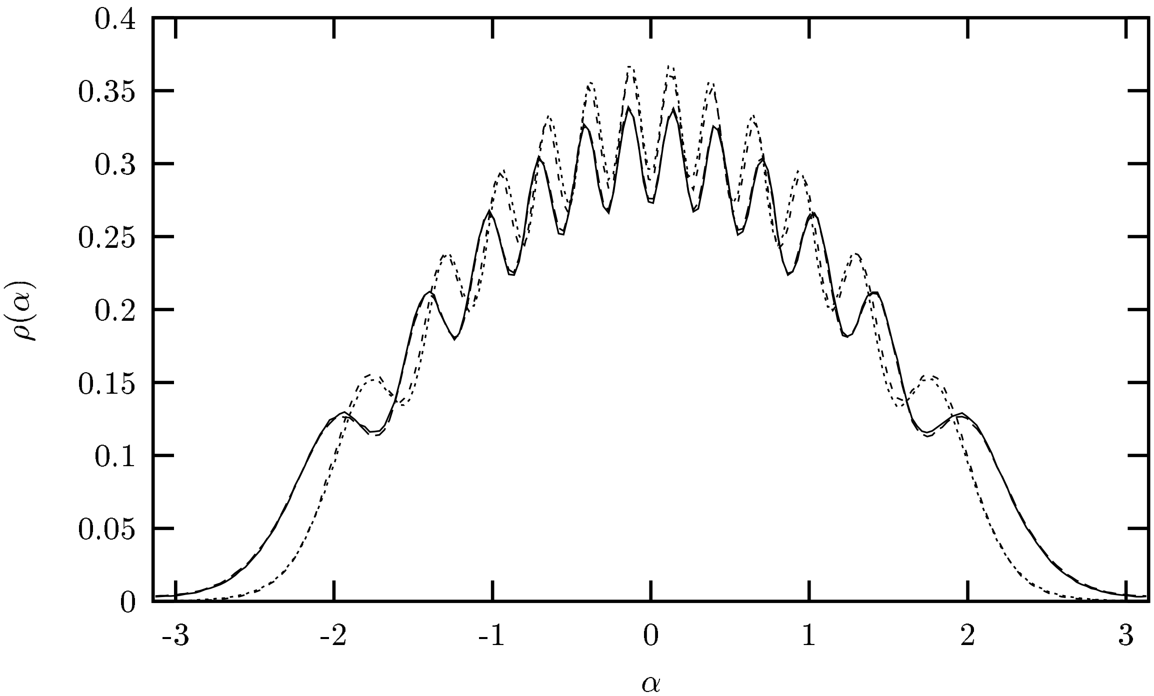
Gross–Witten transition is also a 'local' third order transition. The behaviour of P_3 is remarkably similar:



P_3 in 1+1 dimensions as a function of $\gamma = \frac{\beta}{2N^2}$ for SU(6) (+), SU(12) ($\times$), SU(24) ($*$) and SU(∞) (solid line).

Comparison of eigenvalue spectra

The eigenvalue spectra are also very similar on both sides of the transition:



Density of plaquette eigenvalues for SU(12) in 1+1 dimensions at $\gamma = \frac{\beta}{2N^2} = 0.462$ (long dashes) and $\gamma = 0.542$ (dots) and in 2+1 dimensions at $\gamma = 0.417$ (solid line) and $\gamma = 0.451$ (short dashes).

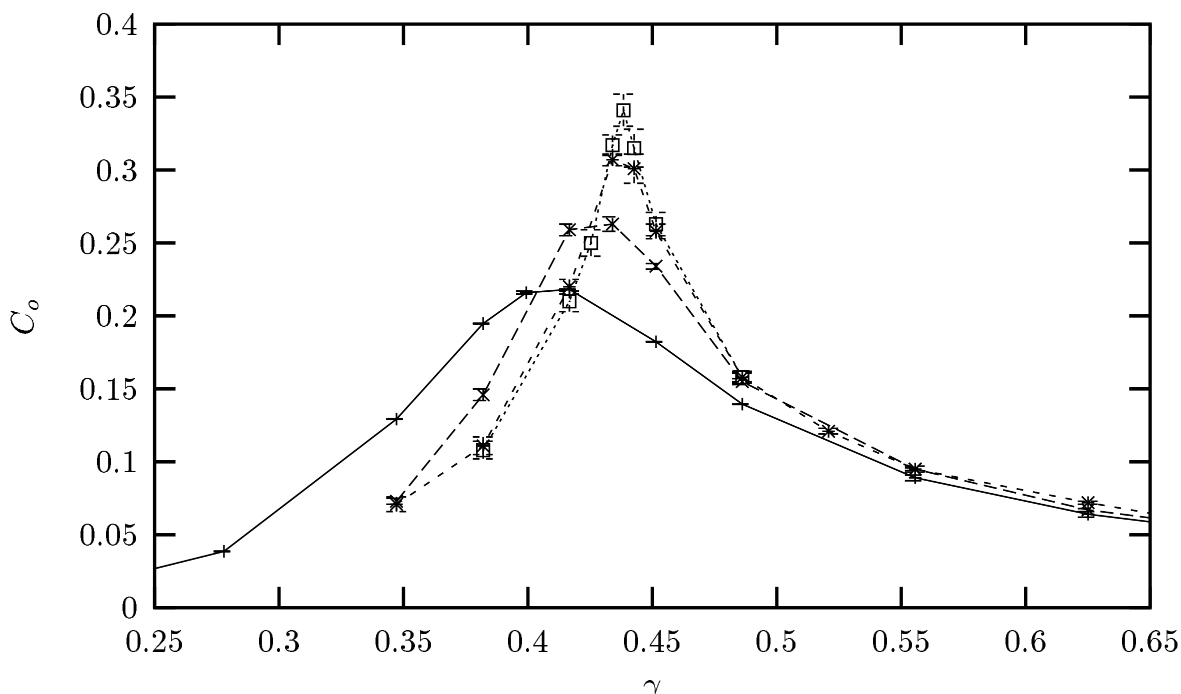
So everything appears to be very similar. Is the bulk transition in $D = 2 + 1$ just a copy of the Gross–Witten transition?

A second order transition?

There is a small peak in the specific heat C_2. This is not present in $D = 1 + 1$.

We see no peak in the local specific heat, P_2. So peak is coming from correlations between plaquettes rather than from fluctuations of individual plaquettes.

We find a growing peak in C_o, the contribution to the specific heat from plaquettes that share an edge:



C_o, as a function of $\gamma = \frac{\beta}{2N^2}$ for SU(6) (+), SU(12) ($\times$), SU(24) ($*$) and SU(48) ($\square$).

Wilson loops

All our results for

- $\langle u_w \rangle$ and its derivative
- The 'local' contribution to the derivative
- The local fluctuations $W_2^{n \times n}$ and $W_3^{n \times n}$

are smooth.

- No sign of transitions becoming sharper for larger N.
- No sign of transitions becoming sharper for larger loops.

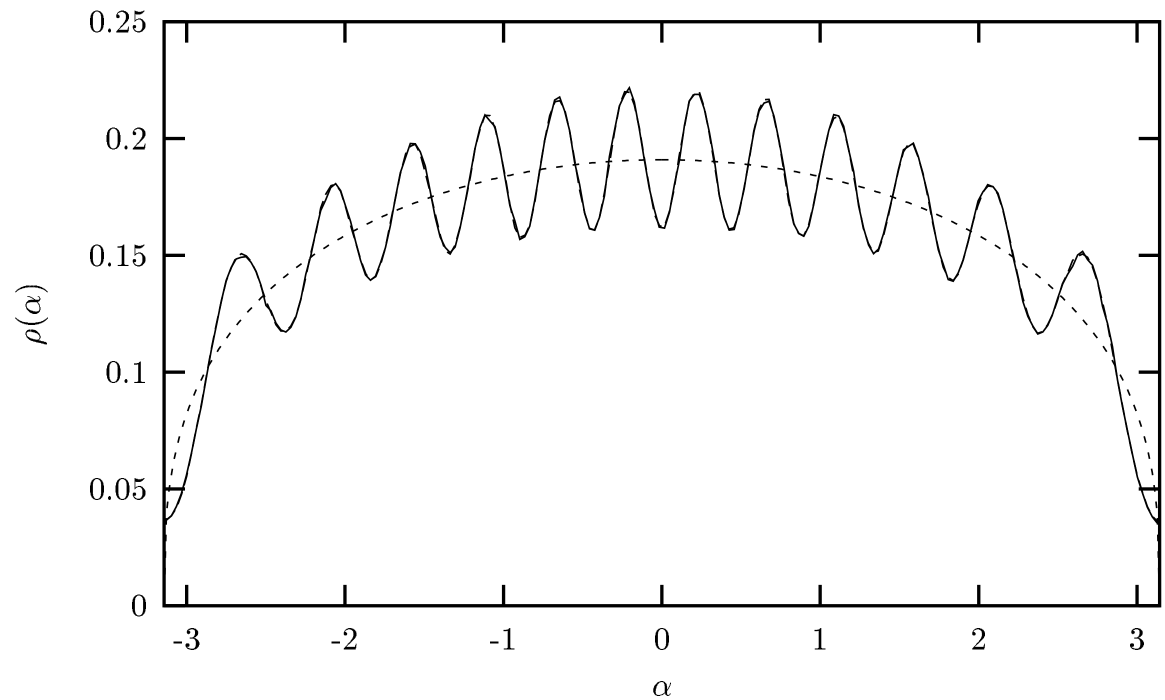
But there is a non–analyticity in $D = 1 + 1$ despite all these quantities being smooth. What about in $D = 2 + 1$?

Look at the gap formation and compare to $D = 1 + 1$.

Matching eigenvalue spectra

In the $D = 1 + 1$ continuum theory, the gap forms when $\langle u_w \rangle = e^{-2}$. We vary the 2+1 dimensional coupling to give the same trace, and then match the two spectra.

E.g. the 3×3 loop in $SU(12)$:



3×3 Wilson loop eigenvalue density, for SU(12) in 1+1 dimensions at $\gamma = \frac{\beta}{2N^2} = 1.255$ (solid line) and in 2+1 dimensions at $\gamma = 0.722$ (long dashes), and the continuum large–N distribution in 1+1 dimensions at $A = A_c$ (short dashes).

Matching eigenvalue spectra

We find it is always possible to achieve this matching

- for any N
- for any loop size

This is convincing evidence that the gap formation is non–analytic in $D = 2 + 1$.

Is gap formation a *continuum* effect?

The trace of the Wilson loop is:

$$\langle u_w \rangle \propto \exp\left\{c\lambda L \log\lambda - c'\lambda^2 L^2\right\} \tag{6}$$

The gap will appear when the trace is e^{-2}. As we approach the continuum limit,

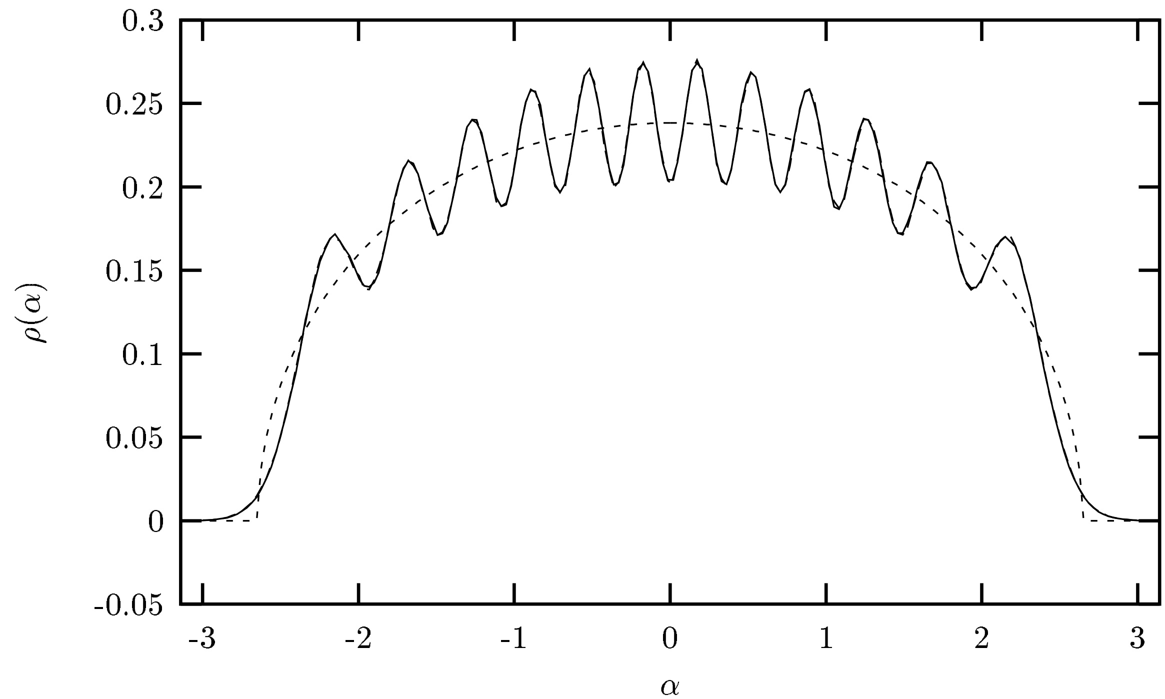
$$A_c \propto \frac{1}{(\log\lambda)^2} \stackrel{a\to 0}{\longrightarrow} 0. \tag{7}$$

So critical area goes into UV in continuum limit.

Matching eigenvalue spectra

Matching of eigenvalue spectra at gap formation is just a special case of a much more general effect: Spectra can *always* be matched.

E.g. matching $D = 1 + 1$ and $D = 2 + 1$ 3×3 loops away from gap formation:



3×3 Wilson loop eigenvalue density for SU(12) in 1+1 dimensions at $\gamma = \frac{\beta}{2N^2} = 2.215$ (solid line) and in 2+1 dimensions at $\gamma = 1.111$ (long dashes), and the continuum large–N distribution in 1+1 dimensions at $A = 0.539 A_c$ (short dashes).

Matching always works

In general:

- Take $n \times n$ loops in D and D' dimensions in $SU(N)$.
- Adjust the couplings so the traces match.

Then the eigenvalue densities will match.

We have tested this for:

- $D = 1 + 1$ and $D = 2 + 1$, and some calculations in $D = 3 + 1$.
- $N = 2$ to $N = 48$.
- 1×1 to 8×8 loops.
- Couplings from $\gamma = 0.25$ to $\gamma = 2.5$

Matching always works

Furthermore,

- 2×2 loops and larger have very similar eigenvalue densities. Rapidly approach continuum limit.
- N dependence is weak

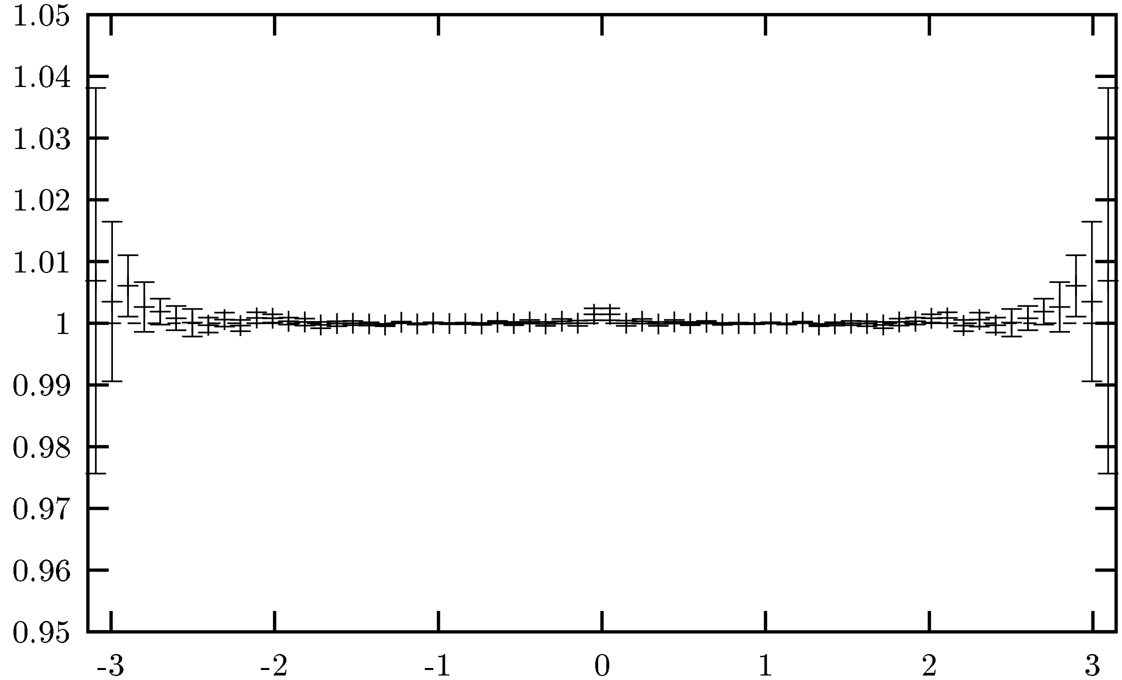
The eigenvalue densities of Polyakov loops can *also* be matched if the traces are matched.

Implies that eigenvalues are not really independent degrees of freedom. They are determined by the trace.

Precision matching

In SU(2), plaquettes do not match exactly.

But for larger loops the differences decrease:



Ratio of 4×4 loop eigenvalue densities in 2+1 and 1+1 dimensions in SU(2) with trace 0.5

Any difference in the continuum limit must be extremely small.

- Fourier components differ by at most 10^{-4}.

We see the same matching in the continuum limit for 3+1d.

Conclusions

There is a bulk transition in 2+1 dimensions.

- A third order phase transition at $N = \infty$
- Very similar to the Gross–Witten transition in $D = 1 + 1$
- There may also be a weak second–order transition developing as $N \to \infty$.

Wilson loops with matching traces have matching eigenvalue densities.

- Not a large N effect.
- Different sizes of loops can also be matched (if large enough).
- Also works for Polyakov loops.

As a consequence of this, Wilson loops in $D = 2 + 1$ have the same non–analyticity as in $D = 1 + 1$.

Conclusions

The eigenvalues of two loops
lie on circles (you could call them hoops),
but here is the catch:
their densities match,
if the traces match of the loops.