Overlap Fermion
in
External Gravity

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Today, only D=4

1. Motivations

Two Success in LGT

1 Gauge Interaction on Lattice

2 Nonperturbative Calculation of Dynamics

Fermion Problems

Two Success in G-W(Overlap) fermion

1 Involvement of Chiral Symmetry

2 Anomaly Calculation （Index）

A Question:

Can we Involve Gravity Interaction and Calculate the Chiral Anomaly on Lattice?

Pioneer: Daemi-Strathdee PRD51('95)

Lattice Gravity $\longrightarrow$ Regge Calculas or Random Lattice

External Gravitational Field $\longrightarrow$ Fixed Lattice (Structure, Spacing)

Our Setting

Lattice $\longrightarrow$ **Fixed "Cubic" Lattice**

No Real Scale, n Just Site Integer

Gravity Int. $\longrightarrow$ Spin Connection $U_{n,\mu} \in \mathrm{Spin}(4)$

Hermiticity (Metric) Condition $\Updownarrow$

Gravity $\longleftarrow$ Vierbein $e^a_\mu(n) \in GL(4, R)$

Continuum Limit

Local Lorentz Invariance $\longrightarrow$ **Exact Lattice Invariance**

General Coordinate Invariance $\longrightarrow$ **Continuum Limit**

Chiral Anomaly Calculation

Is it Trivial? $\longrightarrow$ **No!**

Hermiticity Problem

Spin Connection and Gamma Matrices

$$[\gamma^\mu(n), U_{m,\nu}] \neq 0$$

2. Lattice Formulation with Local Lorentz Invariance

Introducing External Gravitational Field for Dirac Operator

$$\nabla\psi(n) \equiv \sum_{\mu} \gamma^{\mu}(n)(U_{n,\mu}\psi(n+\mu) - \psi(n))$$

$$U_{n,\mu} \in \mathrm{Spin}(4) \qquad U_{n,\mu} = \exp(\frac{1}{2}\gamma^{a}\gamma^{b}\omega_{\mu,ab}(n))$$

$$\gamma^{\mu}(n) \equiv \sum_{a} \gamma^{a} e_{a}^{\mu}(n)$$ ⟵ Inv. Vierbein

Non-Commutativity between Gamma Matrices and Spin Connection

$$[\gamma^{\mu}(n), U_{m,\nu}] \neq 0$$

$\because$

$$\gamma^{a} \in \text{Clifford Algebra of Spin}(4)$$

Local Lorentz Transformation (Gauge Transformation)

$$M^{a}{}_{b}(n) \rightarrow \ g(n) \in \mathrm{Spin}(4)$$

$$e_a^{\mu}(n) \rightarrow (M^{-1}(n))_a{}^b e_b^{\mu}(n)$$

$$\gamma^{\mu}(n) \rightarrow g(n)\gamma^{\mu}(n)g^{-1}(n)$$

$$\psi(n) \rightarrow g(n)\psi(n), \quad \bar{\psi}(n) \rightarrow \bar{\psi}(n)g^{-1}(n)$$

$$U_{n,\mu} \rightarrow g(n)U_{n,\mu}g^{-1}(n+\widehat{\mu})$$

Inner Product

$$(f, g) \equiv \sum_n e(n) f^{T,*}(n) g(n)$$

Weight $e(n) \equiv \det_{a,\mu} e^a_\mu(n)$

$$(f, \nabla g) = -(\nabla^* f, g) - \sum_n \{\sum_\mu \nabla^*_\mu \{e(n)\gamma^\mu(x)\} f(n)\}^{*,T} g(n)$$

Conjugate Difference Operator

$$\nabla^* \psi(n) \equiv \sum_\mu e(n)^{-1} e(n-\widehat{\mu}) U^{-1}_{n-\widehat{\mu},\mu} \gamma^\mu(n-\widehat{\mu}) U_{n-\widehat{\mu},\mu} (\psi(n) - U^{-1}_{n-\widehat{\mu},\mu} \psi(n-\widehat{\mu}))$$

If Yellow Line vanishing, Then

$-\nabla^*$ **Conjugate Operator to** ∇

Hermitian Condition : Metric Condition

$$\sum_\mu \nabla^*_\mu (e(n)\gamma_\mu(n)) = 0$$

$$0 = \sum_\mu \nabla^*_\mu (e(n)\gamma^\mu(n))$$

$$= \sum_\mu \{e(n)\gamma^\mu(n) - e(n-\widehat{\mu})U^{-1}_{n-\widehat{\mu},\mu}\gamma^\mu(n-\widehat{\mu})U_{n-\widehat{\mu},\mu}\}$$

Solving $\Longrightarrow U_{n,\mu}(e^\nu_a(n), e^\nu_a(n+\widehat{\mu}))$

anti-hermitian Operator **hermitian Operator**

$$\nabla + \nabla^*, \quad \nabla\nabla^* + \nabla^*\nabla$$

$$O(\gamma_\mu) \qquad O(1,\ \gamma_\mu\gamma_\nu)$$

Wilson-Dirac Operator

$$D_W = \frac{1}{2}(\nabla + \nabla^*) - \underline{\frac{1}{4}(\nabla\nabla^* + \nabla^*\nabla)}$$

Wilson Term

Hermiticity of Wilson-Dirac Operator

$$\gamma_5 D_W \gamma_5 = D_W^\dagger$$

Overlap-Dirac Operator with External Gravity

$$D_{OD} = 1 - A(A^\dagger A)^{-1/2}$$

$$A \equiv 1 - D_W$$

No Doubling

Ginsparg-Wilson Relation is satisfied

$$\gamma_5 D_{OD} + D_{OD}\gamma_5 = D_{OD}\gamma_5 D_{OD}$$

3. Chiral Anomaly Calculation

Lattice Index

$$\Gamma_5(n,m) \equiv \gamma_5(\delta_{n,m} - \frac{1}{2}D_{OF}(n,m))$$

$$\{\gamma_5 D_{OD}, \Gamma_5\} = 0$$ (Zero mode & Chirality)

$$\widehat{\gamma}_5 \equiv \gamma_5(1 - D_{OD})$$ (Luscher)

$$\delta\psi(n) = i\widehat{\gamma}_5\psi(n) \quad \delta\bar{\psi}(n) = \bar{\psi}(n)i\gamma_5$$

$$\downarrow$$

$$\begin{aligned} \delta J = & -i\sum_n \mathrm{tr}\ (\gamma_5\delta_{n,n} + \widehat{\gamma}_5(n,m)) \\ = & -i2\sum_n \mathrm{tr}\ \Gamma_5(n,n) \end{aligned}$$

Lattice Evaluation of $\delta J = \; -i2\sum_n \text{tr}\ \Gamma_5(n,n)$

Finite Lattice Index: Using Inner Product Definition

$$\sum_n \text{tr}\ \Gamma_5(n,n) = n_+ - n_-$$

zero mode $\gamma_5 D_{OD}$

Deviation from Flat Space

$$h^a_\mu(n) \equiv e^a_\mu(n) - \delta^a_\mu$$

$$\sum_n \text{tr}\ \Gamma_5(n,n) = a^4 \sum_n \mathcal{A}_5(n,n)$$

Estimate of $\mathcal{A}_5(n,n)$

1. After the Classical Continuum Limit, $a \to 0$

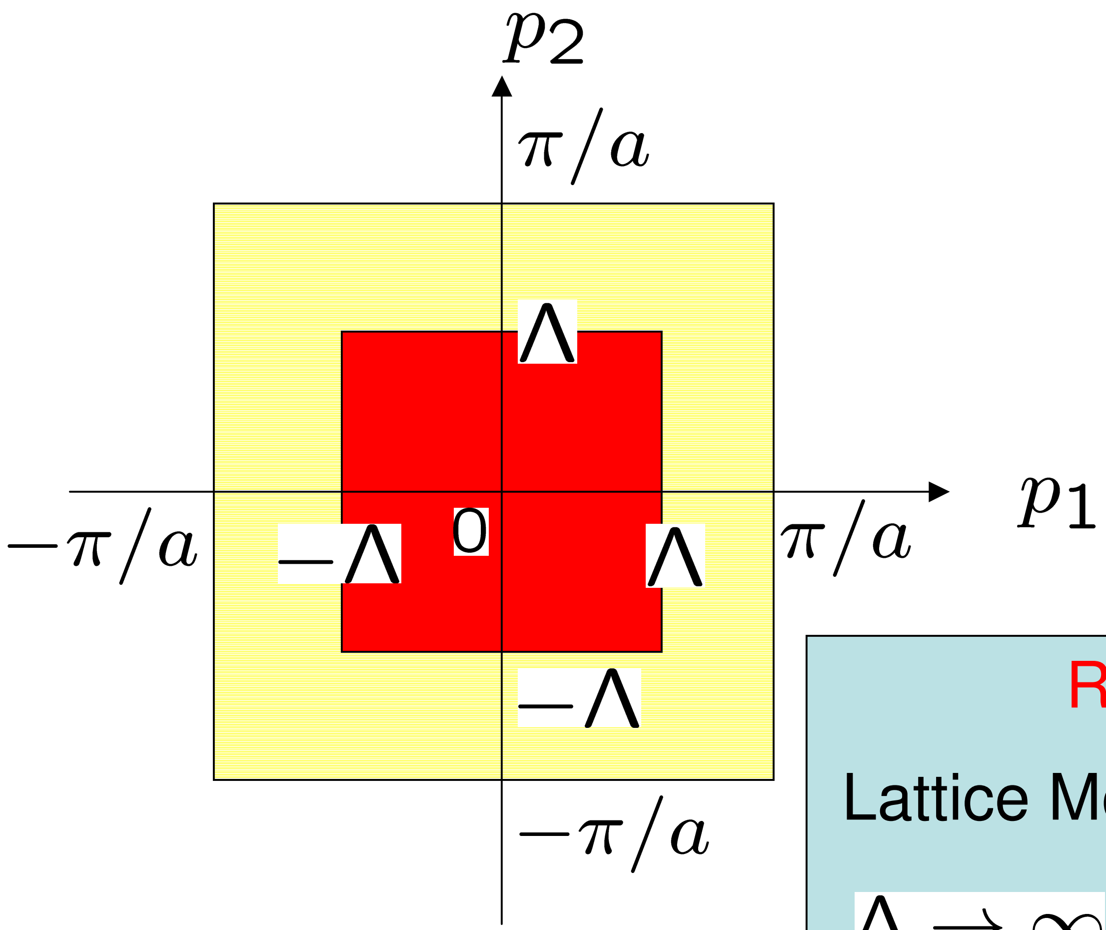
$$U_{n,\mu} = 1 + \frac{1}{2}\gamma^a\gamma^b\omega_{\mu,ab}(n)\cdots$$

(Derivative for External Fields and so on)

2. Keep Lattice Loop Momentum

$$\lim_{a\to 0}\sum_n \mathrm{tr}\ \Gamma_5(n,n) = \ \int_{M^4} \lim_{a\to 0} \mathcal{A}_5(x)$$

Lattice Loop Integral Calculation **D=2**



Yellow Zone Vanishing

After $a \to 0$

Red Zone

Lattice Momenta $\leftrightarrow$ Cont. Momenta

$\Lambda \to \infty$

Dep. on Integrand （Lattice IR Dominant Case Only）

$$\int_{M^4} \lim_{a \to 0} \mathcal{A}_5(x) = \int_{M^4} \det\{\frac{i\hat{R}/4\pi}{\sinh(i\hat{R}/4\pi)}\}^{1/2}$$

Riemann Curvature

$$(\hat{R})^b_a(e) = \frac{1}{2} R^b_{\mu\nu,a}(e) dx^\mu dx^\nu$$

$\det$ About Local Lorentz Indices

4. Summary and Discussions

I Local Lorentz Invariance :
Exact Lattice Symmetry

II General Coordinate Invariance : Restoration
After the Continuum Limit

III Conjugate Op. ∇^* and Hermiticity Condition

IV Wilson-Dirac Operator

V Overlap-Dirac(O-D) Operator and G-W Relation

VI Chiral Index Calculation on Lattice by O-D

Further Study

Majorana Fermion OK!

Another Representations OK

Vector Fields (Link Variable)
Vector-Spinor Fields (Rarita-Schwinger Fields)

Gravity Realization? (Introduction of Scale)

$$< e_\mu^a(n) e_{\nu,a}(n) >= g_{\mu\nu}(n)$$

$$ds^2 \sim g_{\mu\nu}(n) \Delta n_\mu \Delta n_\nu a^2$$