Tadpole and Cactus Improvement of the Gluon Condensate of 3D Yang-Mills Theory

McGill (Montreal, Canada)

Simon Caron-Huot[1]

[1] Department of Physics, McGill University, 3600 University Street, Montreal, Canada H3A 2T8

McGill (Montreal, Canada)

ABSTRACT

We apply the tadpole and cactus improvement methods to the measurement of the gluon condensate of 3D SU(3) Yang-Mills theory. The matching of the lattice measurement to the continuum $\overline{MS}$ scheme involves the subtraction of a series of lattice artifacts divergent up to four loop, which has recently appeared in the literature. The two tested improvement methods are found to significantly increase the convergence to the continuum. Cactus resummation, which is an analytic resummation of a "cactus"-like class of diagrams, is also found to produce a good approximation to the coefficients of this series at a low cost.

1 Introduction

- Recent interest in the gluon condensate of 3D SU(3) Yang-Mills theory

  Related to the non-perturbative $\mathcal{O}(g^6)$ contribution to the thermal QCD free energy

- Pure glue 3D Yang-Mills theory has only one coupling constant g^2, of mass dimension 1 $\Rightarrow$ super-renormalizable

  The theory is well-behaved and perturbative in the UV, but strong-coupled and nonperturbative at a scale of order $k \sim g^2$.

- In the continuum $\overline{\rm MS}$ scheme, for dimensional reasons the gluon condensate must be of the form

$$\langle {\rm Tr}[{\rm F}_{12}^2]\rangle_{\overline{\rm MS}} = {\rm C}g^8 \left[\ln\left(\frac{\mu_{\rm MS}}{2{\rm N_c}g^2}\right) + \beta_{\rm G}\right] \qquad (1)$$

  - The constant C has been computed perturbatively, and arises from the dependence of the plaquette on the introduction of an infrared mass scale [1].
  - However β_G requires non-perturbative input.

$\Rightarrow$ Measure β_G from Lattice Monte-Carlo simulations, using standard Wilson formulation

A problem: the analogous lattice observable $\frac{2N_c}{a^4}\left\langle 1 - \frac{1}{N_c}{\rm Tr}[P_{12}]\right\rangle_a$ is

plagued by divergent lattice artifacts up to four-loop.

Solved in [1] [2] by computing all divergent artifacts using lattice perturbation theory

- Involved a difficult four-loop calculation
- Allows one to take a rigorous limit: ($\beta = 6/g^2a$)

$$\beta_G = \frac{1}{c_4}\lim_{a\to 0}\beta^4\left\{\left\langle 1 - \frac{1}{N_c}{\rm Tr}[P_{12}]\right\rangle_a - \left[\frac{c_1}{\beta} + \frac{c_2}{\beta^2} + \frac{c_3}{\beta^3} + \frac{c_4}{\beta^4}\ln\beta + \frac{c_4'}{\beta^4}\right]\right\} \qquad (2)$$

Our motivation: This limit shows poor convergence

- Large β's required $\Rightarrow$ less than 1 part in 10^5 of the measured signal survives the subtractions.
- Natural to try to apply improvement methods.

This poster will proceed as follow:

- We apply the tadpole improvement method of Lepage and Mackenzie to the data.
- We adapt and apply to our problem the cactus resummation scheme of Panagopoulos and Vicari

A Table of coefficients

We find that the coefficients used when taking the improved limits are:

- Smaller than the unimproved coefficients, whenever they differ
- Very small in the case of the cactus-improved coefficients

coefficient	original	tadpole	cactus	cactus "2-loop"
c_1	8/3	8/3	0	0
c_2	1.951315(2)	1.951315(2)	-0.270907(2)	-0.270907(2)
c_3	6.8612(2)	1.6577(2)	1.7818(2)	0.8929
c_4	2.92942132	2.92942132	2.92942132	2.92942132
c_4'	21.5(9)	-5.0(9)	-0.8(9)	-1.5(9)

Table 1: Comparison of the coefficients used in the different schemes. (The form of the slightly more complicated limit taken in the "cactus 2-loop" case is not given in this poster). There is a certain ambiguity in the coefficient c_4' owing to the choice of the scale appearing inside the logarithm.

References

[1] A. Hietanen, K. Kajantie, M. Laine, K. Rummukainen, and Y. Schröder, JHEP 01 (2005) **013** , hep-lat/0412008

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[3] G.P. Lepage and P.B. Mackenzie, Phys. Rev. D 48 (1993) **5**, hep-lat/9209022

[4] H. Panagopoulos and E. Vicari, Phys. Rev. D58:114501 (1998), hep-lat/9806009

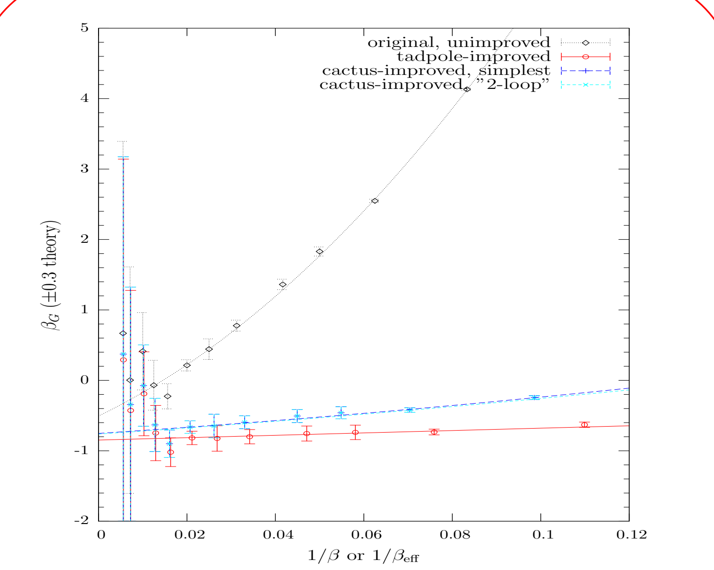


Figure 1: Comparison between the original data of [1] and its improved versions. The fits are quadratic fits in β^{-1}. The interpolation to the continuum limit $1/\beta \to 0$ appears to be much more clean when using the improvement methods.

2 Tadpole improvement

- Try to absorb lattice artifacts through a rescaling of the operators and coupling constant of the theory.
- Motivated by the observation of **Lepage** and **Mackenzie** [3] that the sum average of unitary matrices is not unitary, but is shorter

  Infrared physics should be thought of as small $(1 + iA_\mu)$-type fluctuations occurring on top of the shortened link matrices.

- Estimate this shortening of the infrared field operators in a gauge-invariant fashion by measuring the expectation value of the plaquette

  (The bulk of the plaquette is indeed associated with lattice-scale physics)

One arrives at the simple and well known "tadpole" prescription:

- "Physical" plaquette: $\left(1 - \frac{1}{N_c}{\rm Tr}[{\rm P}_{12}]\right)_{\rm eff} \equiv \left(1 - \frac{1}{N_c}{\rm Tr}[{\rm P}_{12}]\right)/\left\langle\frac{1}{N_c}{\rm Tr}[{\rm P}_{12}]\right\rangle_{\rm a}$
- "Physical" coupling: $\beta_{\rm eff} = \beta\left\langle\frac{1}{N_c}{\rm Tr}[{\rm P}_{12}]\right\rangle_{\rm a}$

We should therefore replace the limit (2) by:

$$\beta_G = \lim_{a\to 0}\frac{\beta\beta_{\rm eff}^3}{c_4^{(T)}}\left\{\left\langle 1 - \frac{1}{N_c}{\rm Tr}[P_{12}]\right\rangle_a - \left[\frac{c_1^{(T)}}{\beta} + \frac{c_2^{(T)}}{\beta\beta_{\rm eff}} + \frac{c_3^{(T)}}{\beta\beta_{\rm eff}^2} + \frac{c_4^{(T)}}{\beta\beta_{\rm eff}^3}\ln\beta_{\rm eff} + \frac{c_4'^{(T)}}{\beta\beta_{\rm eff}^3}\right]\right\} \qquad (3)$$

- Straightforward to obtain the coefficients $c_n^{(T)}$ (The series (2) actually gives us $\beta_{\rm eff}$ in terms of the bare coupling β).

Conclusion: Tadpole **impressively** improves the convergence (Fig.1).

(Question: Beyond reasonable expectation ?)

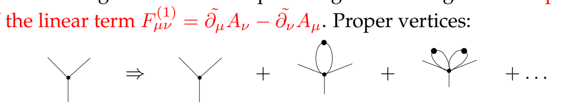
3 Cactus resummation

- Introduced by **Panagopoulos** and **Vicari** [4], as a mean to exploit the observed dominance of tadpole diagrams in lattice perturbation theory.
- An analytic resummation of a certain class of diagrams.

The resummed diagrams can be described as follow. Using the Baker-Campbell-Hausdorff formula, we express the plaquette matrices as a single exponential, which we expand into powers of the gauge field A_μ:

$$P_{\mu\nu} \Rightarrow e^{i(F_{\mu\nu}^{(1)} + F_{\mu\nu}^{(2)} + F_{\mu\nu}^{(3)} + \ldots)} \qquad (4)$$

$\Rightarrow$ The resummed diagrams are the tadpole diagrams arising from the pairwise contractions of the linear term $F_{\mu\nu}^{(1)} = \tilde{\partial}_\mu A_\nu - \tilde{\partial}_\nu A_\mu$. Proper vertices:



(5)

- The blob in the contracted legs indicates that we use the cactus-dressed propagator (e.g. the inverse of the cactus-dressed two-point vertex function, itself dressed like (5).)
- When fully expanded, (5) recursively produces a structure reminiscent of a cactus. (The series of diagram can be understood as solving approximately a Schwinger-Dyson ("gap") equation)

3.1 Cactus-dressing the action

- Calling $g_{\rm eff}^2$ the coefficient of the dressed propagator, the rescaling factor Z of the inverse propagator $g_0^2/g_{\rm eff}^2$ is an analytic function of $g_{\rm eff}^2$ only [4]:

$$Z(g_{\rm eff}^2) \equiv \frac{g_0^2}{g_{\rm eff}^2} = \frac{-6}{N_c^2 - 1}\frac{d}{dg_{\rm eff}^2}G(g_{\rm eff}^2) \qquad (6)$$

  where

$$G(g_{\rm eff}^2) = e^{-g_{\rm eff}^2(N_c-1)/(2dN_c)}L_{N_c-1}^1(g_{\rm eff}^2/3) \qquad (7)$$

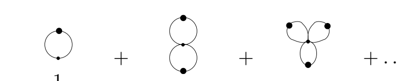
  and $L_{N_c-1}^1$ is a Laguerre polynomial.

  $\Rightarrow$ Solve for $g_{\rm eff}^2$ self-consistently.

- $G(g_{\rm eff}^2)$ is related to the dressing of the unit operator of the plaquette:

$$1 - {\rm Tr}\frac{1}{N_c}P_{12} \Rightarrow 1_{\rm cactus} + \mathcal{O}(A^2) \qquad (8)$$

  where

$1_{\rm cactus} =$ 

$$= 1 - \frac{1}{N_c}G(g_{\rm eff}^2) \qquad (9)$$

- Cactus dressing an operator in the action depends only on the number of $F^{(i)}$'s of (4) out of which it is built.

  All operators of dimension less than 8 (except the unit operator) are rescaled by the same factor Z. Justifies the name $g_{\rm eff}^2 = g_0^2/Z$.

- Dressing of higher dimensional operators is in general more complicated, except when the $2n$ external legs are themselves fully pairwise contracted; then, dressing is a simple rescaling $\propto G^{(n)}(g_{\rm eff}^2)$.

3.2 Implementing cactus resummation

- Systematically, we rewrite the Wilson action using:

$$\frac{1}{g_0^2}\left(1 - \frac{1}{N_c}{\rm Tr}[P_{\mu\nu}]\right) \quad\Rightarrow\quad \frac{1 - \frac{1}{N_c}{\rm Tr}[P_{\mu\nu,{\rm cactus}}]}{g_{\rm eff}^2 Z} + \frac{{\rm Tr}\left[P_{\mu\nu,{\rm cactus}} - P_{\mu\nu}\right]}{g_0^2 N_c} \qquad (10)$$

  We treat the second term as generating higher powers in the loop expansion, e.g. we attach a power of $\hbar$ to each g^2.

- Natural to think of the action as a function of $g_{\rm eff}^2$ only.

  The Z in the first denominator turns out to be approximately canceled by the numerator, for the terms relevant at four loop.

- The resummation (10) is not easy to perform to the four loop level. A much simpler one is obtained by pretending that cactus dressing re-scales everything by Z, except for the term $1_{\rm cactus}$ (which is numerically important and for which this approximation is poor). This replaces the limit (2) by ($\beta_{\rm eff} = 6/g_{\rm eff}^2$):

$$\begin{aligned}\beta_G &= \lim\nolimits_{a\to 0}\tfrac{\beta\beta_{\rm eff}^3}{c_4^{(c)}}\left\{\left\langle 1 - \frac{1}{N_c}{\rm Tr}[P_{12}]\right\rangle_a - 1_{\rm cactus}\right.\\ &\left.\qquad - \left[\frac{c_1^{(c)}}{\beta} + \frac{c_2^{(c)}}{\beta\beta_{\rm eff}} + \frac{c_3^{(c)}}{\beta\beta_{\rm eff}^2} + \frac{c_4^{(c)}}{\beta\beta_{\rm eff}^3}\ln\beta_{\rm eff} + \frac{c_4'^{(c)}}{\beta\beta_{\rm eff}^3}\right]\right\}\end{aligned} \qquad (11)$$

  - The coefficients $c_n^{(c)}$ are straightforward to match with the series (2), since the relation between $g_{\rm eff}^2$ and g_0^2 is known analytically.

  Conclusion: Cactus resummation impressively improves the convergence. It is also seen to account for the bulk of the coefficients (see Table 1).

- One can try to incorporate the effects of the non-trivial rescaling (e.g., $\neq \times Z$) of higher-dimensional operators in a loopwise fashion. At the two loop level, the effects on the coefficients and the fit are indicated in Table 1 and Fig. 1; in particular, the coefficient c_3 is reduced significantly. The author estimated the effect of the corrections associated with such a more systematic treatment, at higher loops, and found that they would change the value of the $\mathcal{O}(\beta^{-2})$ curvature term (and higher terms) on Fig. 1 by a negligible amount, rendering their (nontrivial) calculation inessential.

4 Conclusion

- Both improvement methods were found to impressively improve the convergence to the continuum limit, especially when taking into account the simplicity of their application.
- We have explicitly observed that cactus resummation indeed accounts for the bulk of the coefficients of a four loop series of lattice artifacts.

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