Toward Group Theory Operators for tmLQCD Hadrons

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Abstract

Extraction of the mass spectrum from twisted mass lattice QCD is facilitated by respecting the octahedral group of rotations and accommodating the broken parity and flavor symmetries of the theory. In this work, group theory meson operators adapted to these constraints are constructed for the special case of quark and antiquark fields at a common lattice site, connected by extended gauge field paths.

1 Introduction

1.1 Motivation

- Twisted mass lattice QCD (tmLQCD) offers an efficient mechanism for eliminating unphysical zero modes thereby admitting calculations at lighter quark masses. Automatic $O(a)$ improvement appears at maximal twist.
  [Frezzotti, Grassi, Sint and Weisz, JHEP 08, 058 (2001)]
  [Frezzotti and Rossi, JHEP 08, 007 (2004)]
- Lighter quark mass calculations require improved statistics for hadron mass resolution. (See our other poster regarding the spectrum of tmLQCD.) This fact and the need to disentangle physical states with the same quantum numbers may in part be accomplished through the creation of more operators that represent the channel in question.
- The creation of more elaborate operators also allows for the study of hybrid and exotic mesons.
- The violation of parity by the twisted mass action causes many operators to be twist angle dependent resulting in correlator contamination by opposite parity states when not at maximal twist. This suggests seeking a class of twist independent operators from which the physical parity of an operator is readily deduced.
- Operators with displaced quarks have been constructed using group theoretical techniques but their usage necessarily require the calculation of quark propagators from multiple sources.
  [Basak, Edwards, Fleming, Heller, Morningstar, Richards, Sato, and Wallace, Phys. Rev. D72,094506(2005), Phys. Rev. D72,074501(2005)]

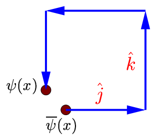
1.2 Lattice Symmetry Group

While parity (P) and charge conjugation (C) may be conserved by lattice actions, the continuous rotational symmetry of nature is broken and one requires operators adapted to the symmetry group of the lattice. For mesons this is the octahedral group O with 24 elements.
For an operator adapted to the representation Λ^{PC}, where $\Lambda \in \{A_1, A_2, E, T_1, T_2\}$ is an irreducible representation of O, one identifies the possible physical states J^{PC} to which it corresponds by the following table which shows the number of copies n^J_Λ of irrep Λ to which the continuum $SU(2)$ irrep J subduces. [Johnson, Phys. Lett. B114, 147(1982)]

J	A_1	A_2	E	T_1	T_2	J	A_1	A_2	E	T_1	T_2
0	1	0	0	0	0	3	0	1	0	1	1
1	0	0	0	1	0	4	1	0	1	1	1
2	0	0	1	0	1	5	0	0	1	2	1

An operator transforming as the irrep E^{+-} could have spin content ($J^{PC} = 2^{+-}, 4^{+-}, 5^{+-}, \ldots$). In theory one needs many operators of each Λ to resolve the many physical spins J in the tower of states to which it corresponds.

1.3 Operator Building Blocks



We construct zero-momentum operators transforming as irreps of O^{PC} from the space of operators spanned by

$$M_{j,k,a,b}(t) = \sum_{\mathbf{x}} \overline{\psi}_a(x) U_j(x) U_k(x+\hat{j}) U_{-j}(x+\hat{j}+\hat{k}) U_{-k}(x+\hat{k}) \psi_b(x) \, ,$$

where $j,k = \pm1, \pm2, \pm3; j \neq k$ and a and b are spinor indices for a total of $24 \times 16 = 384$ operators.
Construction of operators transforming as irrep Λ^{PC} is facilitated by observing that one can effectively consider the transformation of the spinor and link paths independently and then combine them via octahedral Clebsch-Gordan coefficients.

2 Spinor Contribution to Group Structure

The contribution to the group structure due to spinor indices is resolved through consideration of the 16 bilinears $\overline{\psi}F\psi$, where F represents one of sixteen 4×4 matrices, $\{I, \gamma_5, \gamma_4, \gamma_4\gamma_5, \gamma_i, \gamma_i\gamma_5, \sigma_{4i}, \epsilon_{ijk}\sigma_{jk}\}$. The first four bilinears are scalars while the last four three-index objects are vectors under the rotation group in the Euclidean continuum.
Parity (P) and charge conjugation (C) of the the bilinears are identified via

$$\begin{array}{ll} \mathcal{C}\psi\mathcal{C}^\dagger = \left(\overline{\psi}C^\dagger\right)^T & \mathcal{P}\psi\mathcal{P}^\dagger = \gamma_4\psi \\ \mathcal{C}\overline{\psi}\mathcal{C}^\dagger = -\left(C\psi\right)^T & \mathcal{P}\overline{\psi}\mathcal{P}^\dagger = \overline{\psi}\gamma_4 \end{array} \, ,$$

where C is the matrix implementing charge conjugation.
To classify the bilinears into irreps of O one observes that the character table giving $\chi(\xi)$ for O is

$\Lambda\backslash\xi$	E	$3C_4^2$	$8C_3$	$6C_4$	$6C_2$
A_1	1	1	1	1	1
A_2	1	1	1	-1	-1
E	2	2	-1	0	0
T_1	3	-1	0	1	-1
T_2	3	-1	0	-1	1

Since the bilinears are scalars and vectors in the continuum it follows that they are invariant under O. By inspection it is straightforward to show that the character table for the scalar and vector bilinears is identical with that of A_1 and T_1 respectively. It follows trivially by the **representation decomposition formula** for the number n^J_Λ of copies of irrep Λ in the representation J,

$$n^J_\Lambda = \frac{1}{g_O} \sum_\xi p_\xi \chi^{(\Lambda)}(\xi)^* \chi^{(J)}(\xi)$$

that the scalar and vector bilinears form the basis of A_1 and T_1 irreps respectively. Furthermore it may be verified that the i^{th} vector bilinear component transforms as the i^{th} row for our choice of matrix representation $\Gamma^{(T_1)}(R)$.
Hence the reduction of the spinor structure of our operators is

F	Λ^{PC}	λ	α	F	Λ^{PC}	λ	α
I	A_1^{++}	1	1	γ_i	T_1^{--}	i	1
γ_5	A_1^{-+}	1	1	$\gamma_i\gamma_5$	T_1^{++}	i	1
γ_4	A_1^{+-}	1	1	σ_{4i}	T_1^{--}	i	2
$\gamma_4\gamma_5$	A_1^{-+}	1	2	$\epsilon_{ijk}\sigma_{jk}$	T_1^{+-}	i	1

where now each bilinear may be classified uniquely by its irrep Λ^{PC}, row λ, and irrep multiplicity index α.

3 Link Contribution to Group Structure

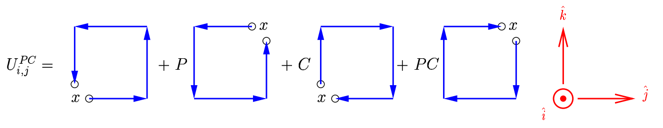
Define the gauge link part of our operators by

$$U_{j,k}(x) \equiv U_j(x) U_k(x+\hat{j}) U_{-j}(x+\hat{j}+\hat{k}) U_{-k}(x+\hat{k}) \, ,$$

where $j,k = \pm1, \pm2, \pm3; j \neq k$.
The parity and charge conjugation contributions due to the link structure can be taken into account by defining the PC-adapted superpositions,

$$U^{PC}_{i,j} = U_{j,k} + PU_{-j,-k} + CU_{k,j} + PCU_{-k,-j} \, ,$$

where k is defined via $\hat{k} = \hat{i} \times \hat{j}$. Diagrammatically, one has



For fixed PC the space spanned by $U^{PC}_{i,j}$ is invariant under O and will generate a representation U^{PC} of the group. Restricting (i,j) to $\{(1,2),(1,3),(2,3),(2,1),(3,1),(3,2)\}$ gives a basis for the six-dimensional space and will provide the particular matrix representation $\Gamma^{(U^{PC})}_{(i,j)(m,n)}(R)$. The character table for U^{PC} giving $\chi^{(U^{PC})}(\xi)$ is

E	$3C_4^2$	$8C_3$	$6C_4$	$6C_2$
6	$2P$	0	0	$(P+1)C$

Use of the representation reduction formula reduces U^{PC} to irreps Λ^{PC} of O^{PC} as follows

P	C	A_1^{PC}	A_2^{PC}	E^{PC}	T_1^{PC}	T_2^{PC}
+	+	1	0	1	0	1
+	−	0	1	1	1	0
−	+	0	0	0	1	1
−	−	0	0	0	1	1

The symmetrized link fields corresponding to this reduction are

$$U^{\Lambda^{PC}}_\lambda(x) = \sum_{(i,j)} \frac{\sum_{R\in O} \Gamma^{(U^{PC})}_{(i,j)(m,n)}(R) \Gamma^{(\Lambda)}_{\lambda\mu}(R)^*}{\left[\frac{g_O}{d_\Lambda} \sum_{R\in O} \Gamma^{(U^{PC})}_{(m,n)(m,n)}(R) \Gamma^{(\Lambda)}_{\mu\mu}(R)^*\right]^{\frac{1}{2}}} U^{PC}_{i,j}(x) \, ,$$

where (m,n) and μ are chosen so the denominator does not vanish and the accessible Λ^{PC} are chosen from the above table. Here $\Gamma^{(\Lambda)}_{\lambda\mu}(R)$ is our matrix representation for the irrep Λ. The $U^{\Lambda^{PC}}_\lambda(x)$ transform as the row λ of irrep Λ^{PC} and are uniquely identified by them.

4 Total Representation Reduction and Operator Construction

4.1 Reduction of Direct Product Representations

Having classified the spinor and link components of our operators into irreps of O^{PC}, it remains to combine them and reduce all possible direct product representations. Parity and charge conjugation for the product representations satisfy $P = P_f P_u$ and $C = C_f C_u$. Noting that the character of $\Lambda_1 \otimes \Lambda_2$ satisfies $\chi^{(\Lambda_1\otimes\Lambda_2)}(\xi) = \chi^{(\Lambda_1)}(\xi)\chi^{(\Lambda_2)}(\xi)$, and using the representation decomposition formula one may reduce each direct product into irreps of O. As such the 384-dimensional representation generated by the space spanned by $M_{j,k,a,b}(t)$ may be reduced into irreps of O^{PC} as follows

$\otimes$	A_1	A_2	E	T_1	T_2
A_1	A_1	A_2	E	T_1	T_2
T_1	T_1	T_2	$T_1 \oplus T_2$	$A_1 \oplus E \oplus T_1 \oplus T_2$	$A_2 \oplus E \oplus T_1 \oplus T_2$

$\Longrightarrow$

$\Lambda\backslash PC$	$++$	$+-$	$-+$	$--$
A_1	4	4	6	2
A_2	4	4	2	6
E	8	8	8	8
T_1	12	12	10	14
T_2	12	12	14	10

Note that all possible lattice irreps are accessible, not just the A_1^{PC} and T_1^{PC} of simpler local operators.

4.2 Operator Construction Using Clebsch-Gordan Coefficients

The construction of the symmetrized operators themselves is accomplished by combining the spinor and link operator components $F^{\Lambda^{PC},\alpha}_\lambda$ and $U^{\Lambda^{PC}}_\lambda$ using the same formulae for parity and charge conjugation given above and by combining the irreps of O using Clebsch-Gordan coefficients for $\Lambda_1 \otimes \Lambda_2$. These are given by the following formula allowing the creation of symmetrized operators $M^{\Lambda^{PC}}_\lambda(t)$,

$$\left(\begin{array}{cc|c} \Lambda_1 & \Lambda_2 & \Lambda \\ \lambda_1 & \lambda_2 & \lambda \end{array}\right) = \frac{\sum_{R\in O} \Gamma^{(\Lambda_1)}_{\lambda_1\mu_1}(R) \Gamma^{(\Lambda_2)}_{\lambda_2\mu_2}(R) \Gamma^{(\Lambda)}_{\lambda\mu}(R)^*}{\left[\frac{g_O}{d_\Lambda} \sum_{R\in O} \Gamma^{(\Lambda_1)}_{\mu_1\mu_1}(R) \Gamma^{(\Lambda_2)}_{\mu_2\mu_2}(R) \Gamma^{(\Lambda)}_{\mu\mu}(R)^*\right]^{\frac{1}{2}}} \Longrightarrow M^{\Lambda^{PC},\Lambda_f^{P_fC_f},\alpha_f,\Lambda_u^{P_uC_u}}_\lambda(t) = \sum_{\mathbf{x}} \sum_{\lambda_f,\lambda_u} \left(\begin{array}{cc|c} \Lambda_f & \Lambda_u & \Lambda \\ \lambda_f & \lambda_u & \lambda \end{array}\right) \overline{\psi}(x) F^{\Lambda_f^{P_fC_f},\alpha_f}_{\lambda_f} U^{\Lambda_u^{P_uC_u}}_{\lambda_u}(x) \psi(x) \, .$$

In the C-G coefficient formula μ, μ_1, and μ_2 are chosen so the denominator does not vanish. In the operator formula the allowed irreps for the spinor bilinear and link components are determined in sections 3 and 4. These fix P and C for the operator while the irrep Λ of O are those allowed by the C-G series table in the previous section. Dirac indices on F and color indices on U and both indices on the spinors have been suppressed.

5 And Now for the Twist...

5.1 Operators with Definite Parity

Consider tmLQCD for a quark doublet.
A change of basis (by twist angle ω) gives

$$\left(\begin{array}{c} u' \\ d' \end{array}\right) = \exp\left(i\omega\gamma_5\tau_3\right) \left(\begin{array}{c} u \\ d \end{array}\right) \, ,$$

which leads to mixing among some, but not all, quark bilinears.
For example,

$$\bar{u}'\gamma_i d' = \bar{u}\gamma_i d \cos\omega + \bar{u}\gamma_i\gamma_5 d \sin\omega$$

couples to both vector and axial mesons unless ω is carefully tuned, but

$$\bar{u}'\sigma_{4i}d' = \bar{u}\sigma_{4i}d$$

couples to the vector and never to the axial.

Explicit calculation shows that tmLQCD splits the results of section 2 into two sets, according to flavor:
red structures are only independent of twist angle for charged mesons,
green structures are only independent of twist angle for neutral mesons.

Despite these restrictions, one finds that *all Λ^{PC} combinations can still be obtained by factoring in some nontrivial link structure.*

charged mesons*

$\Lambda\backslash PC$	++	+-	-+	−
A_1	3	1	3	1
A_2	1	3	1	3
E	4	4	4	4
T_1	5	7	5	7
T_2	7	5	7	5

neutral mesons

$\Lambda\backslash PC$	++	+-	-+	−
A_1	1	3	3	1
A_2	3	1	1	3
E	4	4	4	4
T_1	7	5	5	7
T_2	5	7	7	5

5.2 Preserving Parity in the Correlators

Building states of definite Λ^{PC} might seem insufficient for tmLQCD, since the theory does not preserve P. However, tmLQCD does conserve a “twisted parity”, $\tilde{P}$ defined by the product of P and ($\omega \to -\omega$). Therefore, states of definite twisted parity are automatically preserved in any correlation function. Since our quark bilinears are *independent* of ω, P and $\tilde{P}$ are equivalent for these bilinears.

*Note that the charged mesons are really only eigenstates of g-parity with eigenvalue $G = C(-1)^I$, but the argument may be corrected in a straightforward manner.