Improved Heavy Quark Action

Norman H. Christ, Min Li and Huey-Wen Lin

Columbia University

RBC and UKQCD Collaborations

Tucson — July 24, 2006

OUTLINE

- **Introduction.**
- **Symanzik improvement applied to heavy quarks.**
- **On-shell improvement.**
- **On-shell spinor matrix elements.**
- **Tree-level check.**
- **Conclusions.**

Introduction

- Charm and bottom quarks are too heavy for current lattice ensembles: $m \sim 1/a$.

- Difficulty overcome by:
  - Heavy quark effective theory (HQET).
  - Non-relativistic QCD (NRQCD).
  - Relativistic Heavy Quarks / Fermilab (RHQ).

- We will adopt RHQ approach.
  - Works for all lattice spacings $a \ll 1/m$ and $a \gg 1/m$.
  - Allows continuum limit.
  - Supports non-perturbative methods.

Symanzik Improvement and Heavy Quarks

- **Begin with a standard lattice Wilson fermion action:**

$$S_{\rm lat} = \sum_{n',n} \overline{\psi}_{n'} \left(\gamma^\mu D^\mu + m_0 - \frac{r}{2}(D^\mu)^2 + \sum_{i,j} \frac{i}{4} c_{SW} \sigma_{\mu\nu} F_{\mu\nu} \right)_{n',n} \psi_n,$$

where

$$\begin{aligned} (D_\mu \psi)_n &= \frac{1}{2}\left[U_\mu(n)\psi_{n+\hat\mu} - U_\mu(n-\hat\mu)^\dagger \psi_{n-\hat\mu}\right] \\ (D^2_\mu \psi)_n &= \left[U_\mu(n)\psi_{n+\hat\mu} + U_\mu(n-\hat\mu)^\dagger \psi_{n-\hat\mu} - 2\psi_n\right] \\ (F_{\mu\nu}\psi)_n &= \frac{1}{8} \sum_{s,s'=\pm 1} s s' \left[U_{s\mu}(n) U_{s'\nu}(n+s\hat\mu)\right. \\ &\quad \left. \times U_{-s\mu}(n+s\hat\mu+s'\hat\nu) U_{-s'\nu}(n+s'\hat\nu) - \mathrm{h.c.}\right] \psi_n. \end{aligned}$$

- **Long distance physics given by the Symanzik effective action:**

$$S_{\rm eff} = \int d^4x \, \overline{\psi}(x) \left(\gamma^\mu D^\mu + m_r - a\frac{r^c}{2}(D^\mu)^2 + a \sum_{\mu,\nu} \frac{i}{4} c^c \sigma_{\mu\nu} F_{\mu\nu} \right) \psi(x).$$

Relativistic Heavy Quarks

- **Work in the rest system with $|\vec{p}a| \ll 1$.**
- **Include higher orders in $(ma)^n$ and $(D_0 a)^{2l}$ in S_{eff}:**

$$\mathcal{L}_{\mathrm{eff}} = \mathcal{L}_{\mathrm{eff},-1} + \mathcal{L}_{\mathrm{eff},0} + \mathcal{L}_{\mathrm{eff},1} + \ldots$$

where

$$\begin{aligned}
\mathcal{L}_{\mathrm{eff},-1} &= \overline{\psi}\left(\frac{1}{a}B^{-1,1} + \gamma^0 D^0 C^{-1,1}\right)\psi \\
\mathcal{L}_{\mathrm{eff},0} &= \overline{\psi}\left(\{\vec{\gamma}\vec{D}, B^{0,1}\} + a\{[\vec{\gamma}\vec{D}, \gamma^0 D^0], C^{0,1}\}\right)\psi \\
\mathcal{L}_{\mathrm{eff},1} &= a\overline{\psi}\Big(\vec{D}^2 B^{1,1} + a\{\vec{D}^2, \gamma^0 D^0\}C^{1,1} \\
&\quad + [\gamma^i, \gamma^j][D^i, D^j]B^{1,2} + a\{[\gamma^i, \gamma^j][D^i, D^j], \gamma^0 D^0\}C^{1,2} \\
&\quad + [\gamma^i, \gamma^0][D^i, D^0]B^{1,3} + a[[\gamma^i, \gamma^0][D^i, D^0], \gamma^0 D^0]C^{1,3}\Big)\psi.
\end{aligned}$$

- **Coefficients $B^{i,j}$ and $C^{i,j}$ are polymonial in $m_0 a$, $(D^0 a)^2$ and g^2.**
- **Treat $m_0 a$ and $D_0 a$ as $O(1)$.**

Field Transformations

- **Consider field transformations of $O(1)$, $O(a^1)$ and $O(a^2)$:**

$$\begin{aligned}
O(1):\quad \psi &= \left(1 + R^{0,1} + a\gamma^0 D^0 S^{0,1}\right)\psi' \\
\overline{\psi} &= \overline{\psi}'\left(1 + R^{0,1} - a\gamma^0 \overleftarrow{D}^0 S^{0,1}\right)
\end{aligned}$$

$$\begin{aligned}
O(a):\quad \psi &= \psi'\left(1 + a\vec{\gamma}\vec{D}R^{1,1} + a[\vec{\gamma}\vec{D}, a\gamma^0 D^0]S^{1,1}\right)\psi' \\
\overline{\psi} &= \overline{\psi}'\left(1 - aR^{1,1}\vec{\gamma}\overleftarrow{D} - aS^{1,1}[\vec{\gamma}\overleftarrow{D}, a\gamma^0\overleftarrow{D}^0]\right)
\end{aligned}$$

$$\begin{aligned}
O(a^2):\quad \psi &= \Big(1 + a^2\vec{D}^2 R^{2,1} + a^2\{\vec{D}^2, a\gamma^0 D^0\}S^{2,1} \\
&\quad + a^2[\gamma^i,\gamma^j][D^i,D^j]R^{2,2} + a^2\{[\gamma^i,\gamma^j][D^i,D^j], a\gamma^0 D^0\}S^{2,2} \\
&\quad + a^2[\gamma^i,\gamma^0][D^i,D^0]R^{2,3} + a^2[[\gamma^i,\gamma^0][D^i,D^0], a\gamma^0 D^0]S^{2,3}\Big)\psi' \\
\overline{\psi} &= \overline{\psi}'\Big(1 + a^2\overleftarrow{D}^2 R^{2,1} - a^2\{\overleftarrow{D}^2, a\gamma^0\overleftarrow{D}^0\}S^{2,1} \\
&\quad + a^2[\gamma^i,\gamma^j][\overleftarrow{D}^i,\overleftarrow{D}^j]R^{2,2} - a^2\{[\gamma^i,\gamma^j][\overleftarrow{D}^i,\overleftarrow{D}^j], a\gamma^0\overleftarrow{D}^0\}S^{2,2} \\
&\quad + a^2[\gamma^i,\gamma^0][\overleftarrow{D}^i,\overleftarrow{D}^0]R^{2,3} + a^2[[\gamma^i,\gamma^0][\overleftarrow{D}^i,\overleftarrow{D}^0], a\gamma^0\overleftarrow{D}^0]S^{2,3}\Big).
\end{aligned}$$

Field Transformations (continued)

- Chose coefficients R^{ij} and S^{ij} to simplify $\mathcal{L}_{\rm eff}$.

- Linearize by working at a fixed $N = k + l + n$ with terms of order $(m_0 a)^k (D_0 a)^{2l} g^{2n}$.

- Carry out induction in N.

- The effective action can be simplified to the form:

$$S_{\rm eff} \;=\; \int d^4x\; \overline{\psi}(x) \left(\gamma^0 D^0 + \zeta^c \vec{\gamma} \cdot \vec{D} + m_r + \sum_{\mu,\nu} \frac{i}{4} c_P^c a \sigma_{\mu\nu} F_{\mu\nu} \right) \psi(x).$$

- Tune the lattice action to set $m_r = m_{\rm heavy}$, $\zeta^c = 1$ and $c_P^c = 0$.

- This permits a 3-parameter lattice action:

$$S_{\rm lat} \;=\; \sum_{n',n} \overline{\psi}_{n'} \left(\gamma^0 D^0 + \zeta \vec{\gamma} \cdot \vec{D} + m_0 - \frac{1}{2}(D^0)^2 - \frac{\zeta}{2}\vec{D}^2 + \sum_{\mu,\nu} \frac{i}{4} c_P \sigma_{\mu\nu} F_{\mu\nu} \right)_{n',n} \psi_n.$$

- Tune m_0, ζ and c_P.

What happened to $c_E \neq c_B$?

- Removed by $O(a^2)$ transformation acting on $O(1/a)$ term in action:

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \overline{\psi}(x)\left(1+\alpha a^2[\gamma^i,\gamma^0][D^i,D^0]\right)\\ &\quad\cdot\left(\gamma^0 D^0+m_r\right)\left(1+\alpha a^2[\gamma^i,\gamma^0][D^i,D^0]\right)\psi(x)\\ &= \overline{\psi}(x)\left(\gamma^0 D^0+m_r+\alpha 2m_r a^2[\gamma^i,\gamma^0][D^i,D^0]\right.\\ &\quad\left.+\alpha a^2\{[\gamma^i,\gamma^0][D^i,D^0],\gamma^0 D^0\}\right)\psi(x).\end{aligned}$$

- The $\alpha 2m_r a^2$ term can remove the difference between c_E and c_B which is proportional to $m_r a$.

- The second α term $\sim \alpha a^2\big[[D^i,D^0],D^0\big] \sim a^2$ and can be dropped.

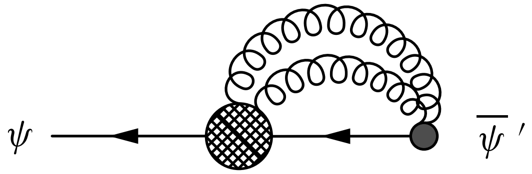
- Attempt to remove $m_r a$ dependence of c_B using $\beta a^2[\gamma^i,\gamma^j][D^i,D^j]$ fails because now the second term $\sim \beta a^2\big\{[D^i,D^0],D^0\big\} \sim a$.

Implications for on-shell improvement

- Invariance under field transformations implies the particle spectrum of the tuned lattice theory and the continuum action will agree to $O(|\vec{p}a|)$, $O(ma)^n$ and $O(|\vec{p}a|(ma)^n)$.
- However, on-shell spinor lattice Greens functions will not agree! (A point emphasized by the Tsukuba group.)
- The field transformations depend on the unknown redundant parameters and can't actually be performed.
- On-shell the bare and transformed fields will differ by a non-covariant, spinor Z-factor:

$$\psi_\alpha(x) \ = \ Z_{\alpha,\beta}\,\psi'_\beta(x) = \frac{1}{Z_q}\left(1 - i\,\delta_\psi \vec{\gamma}\cdot\vec{\nabla}\right)_{\alpha,\beta}\psi'_\beta(x).$$

- On-shell, the non-linear terms collapse:



Tree-level check

- **Begin with the general anisotropic lattice Wilson action:**

$$\begin{aligned} S_{\text{lat}} &= \sum_{n',n} \overline{\psi}_{n'} \Big(\gamma^0 D^0 + \zeta \vec{\gamma} \cdot \vec{D} + m_0 - \frac{r_t}{2} (D^0)^2 - \frac{r_s}{2} \vec{D}^2 \\ &\quad + \sum_{i,j} \frac{i}{4} c_B \sigma_{ij} F_{ij} + \sum_i \frac{i}{2} c_E \sigma_{0i} F_{0i} \Big)_{n',n} \psi_n \end{aligned}$$

- **Tree-level quark propagator takes the continuum form if:**

$$\begin{aligned} m_r a &= \ln \left(\frac{m_0 a + r_t + \sqrt{(m_0 a)^2 + 2 r_t m_0 a + 1}}{1 + r_t} \right) \\ Z_q &= \cosh(m_r a) + r_t \sinh(m_r a) \\ \zeta^2 &= \frac{\sinh(m_r a)}{m_r a} (r_t \sinh(m_r a) + \cosh(m_r a)) - r_s \sinh(m_r a). \\ \delta_\psi &= \frac{\zeta}{2 \sinh(m_r a)} - \frac{1}{2 m_r a} \end{aligned}$$

- **Similarly, the quark-gluon vertex contains no errors of $O(|\vec{p}a|)$, $O(ma)^n$ or $O(|\vec{p}a|(ma)^n)$ for $c_B = c_E = r_s$.**
- **All $(\vec{p}a)^2$ errors are bounded for $0 \leq m_r a < \infty$.**

Conclusions

- Removing error of order $|\vec{p}a|$ and all orders in $(ma)^n$ requires knowing three functions: $m_0(m_r a)$, $\zeta(m_r a)$ and $c_P(m_r a)$.
  - m_0 must come for experiment.
  - ζ is easy to determine from $E(\vec{p}) = m_r + \frac{\vec{p}^2}{2m_r}$.
  - $c_P(m_r a)$:
    - $*$ Fit to experimental hyperfine splitting.
    - $*$ Determine from step scaling proceedure.
  - See HueyWen Lin's talk for results from both approaches!
- Spinor Green's functions require a fourth function $\delta_\Psi(m_r a)$.
  - Need δ_Ψ for specific composite operators Ψ, *e.g.* the lattice operator which creates a charmed baryon.
  - Easily determined from spin-1/2 state propagators.
- Such non-perturbative approaches should provide an important alternative to tree-level or one-loop coefficients.