Definition and parametrization of non-perturbative effects in quenched QCD

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The Main Questions I Want to Address:

- Can define "the nonperturbative part" of the average plaquette in a consistent way? The problem is that perturbative series are usually diverging and scheme dependent.

- If yes, can we parametrize it in terms of expressions of the form $Ag^B e^{-\frac{C}{g^2}}$? Hopefully, A, B and C could be calculable semi-classically.

Content of the talk

- The rule of thumb (at a given coupling, you drop the order in perturbation theory that gives the smallest contribution and all the higher orders). This is a poor man substitute for regularizing the diverging perturbative series by introducing a large field cutoff or a large action cutoff.
- The double-well. A QM example where the methods works.
- QCD with one plaquette. Another example where everything works well.
- Quenched QCD. Λ^{A}_{2-loop} versus a^{B} form; complex singularities; large order extrapolations; parametrization of $a(\beta)$.

The rule of thumb for divergent series

Drop the order with the smallest contribution (and all higher orders)

Take a generic asymptotic series: $A \sim \sum_k a_k \lambda^k$

Error at order $k \equiv \Delta_k(\lambda) = A_{numerical}(\lambda) - \sum_{l=0}^{k} a_l \lambda^l$

We assume that $\Delta_k \simeq \lambda^{k+1} a_{k+1}$ (for λ small enough)

Large order behavior: $|a_k| \sim |C_1||C_2|^k \Gamma(k + C_3)$

The error is minimized for $k^\star \simeq (\lambda |C_2|)^{-1} - C_3 - (1/2) + \mathcal{O}(1/k^\star)$

$Min_k\ |\Delta_k| \simeq \sqrt{2\pi}|C_1|(\lambda|C_2|)^{1/2-C_3} \mathrm{e}^{-\frac{1}{|C_2|\lambda}}$ (order independent)

Accuracy curves and their perturbative envelope

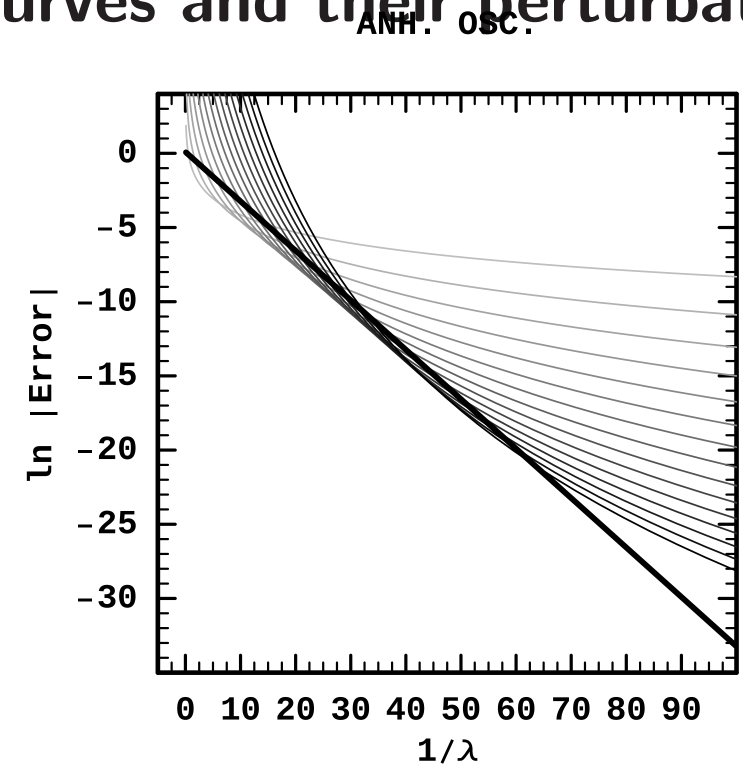

Figure 1: ln($|Error|$) for order 1 to 15 for the anharmonic oscillator as a function of $1/\lambda$. As the order increases, the curves get darker. The thicker dark curve is ln $((\sqrt{12}/\pi)\mathrm{e}^{-\frac{1}{3\lambda}})$

The double-well

Potential: $V(y) = (1/2)y^2 - gy^3 + (g^2/2)y^4$

Ground state energy: $E_0 \sim \sum_{k=0} a_k \; g^{2k}$

Asymptotic behavior (Brezin et al.): $a_k \sim -(3/\pi)3^k\Gamma(k+1)$

Perturbative envelope: $Min_k \; |\Delta_k| \simeq \sqrt{6/\pi} g^{-1} \mathrm{e}^{-\frac{1}{3g^2}}$

1-instanton effect: $\Delta E_0 = -(g\pi)^{-1/2} \mathrm{e}^{-\frac{1}{6g^2}}$

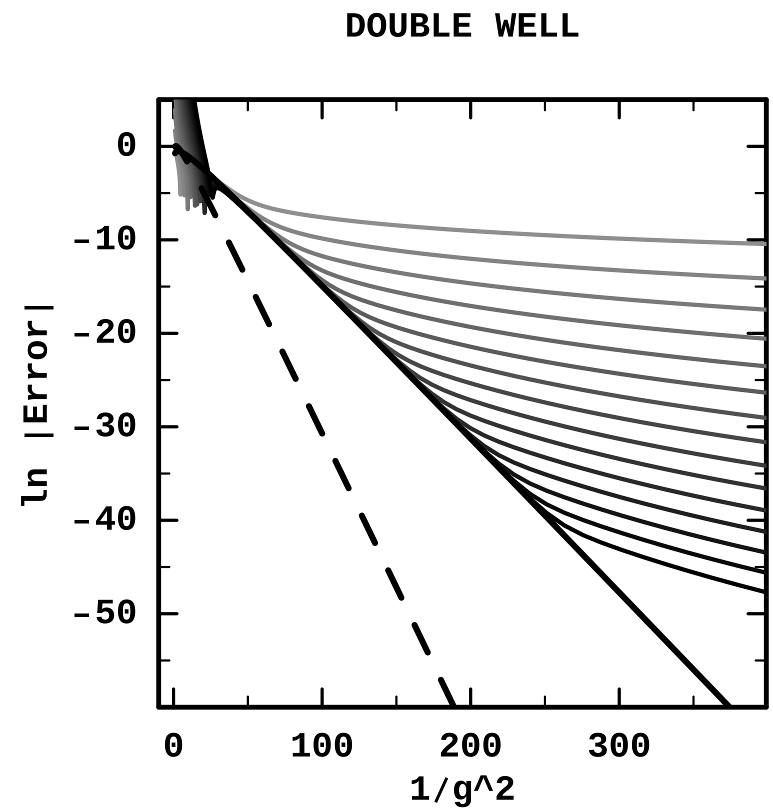

Figure 2: ln($|Error|$) for order 1 to 15 (in g^2) versus $1/g^2$ for the ground state of the double-well potential. As the order increases, the curves get darker. The thicker dark curve is ln($(g\pi)^{-1/2}e^{-\frac{1}{6g^2}}$) (1-instanton). The dash curve is ln $(\sqrt{6/\pi}g^{-1}e^{-\frac{1}{3g^2}})$.

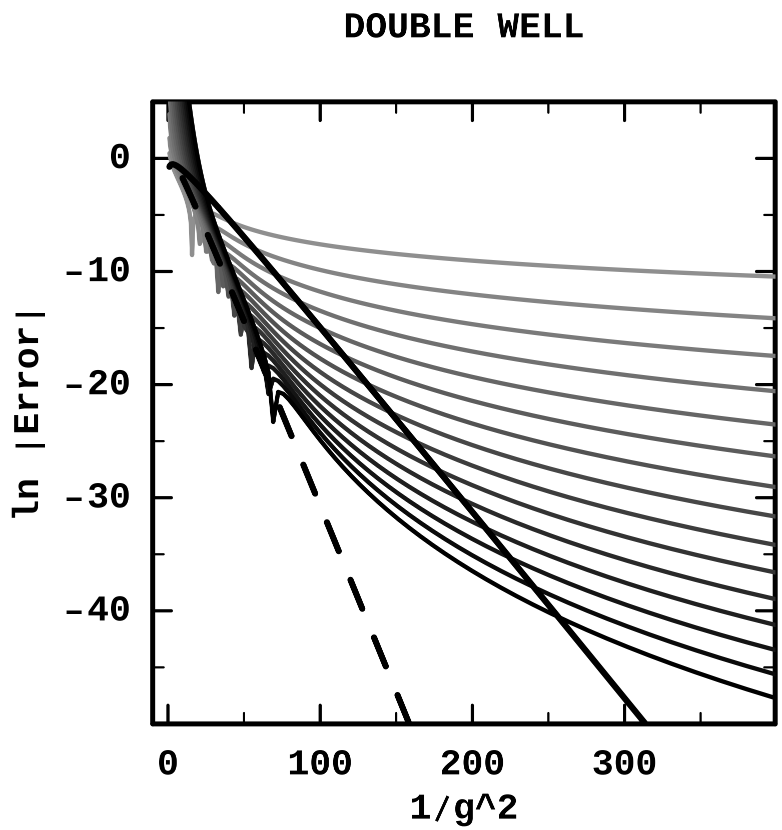

Figure 3: $\ln(|Error|)$ for order 1 to 15 (in g^2) versus $1/g^2$ for the average between the two lowest energy states of the double-well potential. As the order increases, the curves get darker. The thicker dark curve is $(g\pi)^{-1/2}\mathrm{e}^{-\frac{1}{6g^2}}$. The dash curve is $\sqrt{6/\pi}g^{-1}\mathrm{e}^{-\frac{1}{3g^2}}$ (minimal error).

1 plaquette LGT, (Li, YM PRD 71 054509 (2005))

$$Z(\beta, N) = \int \prod_{l \in p} dU_l \mathrm{e}^{-\beta(1-\frac{1}{N}ReTrU_p)} \ ,$$

$$Z(\beta, 2) = (2/\beta)^{3/2}\frac{1}{\pi}\int_0^{2\beta} dt t^{1/2}\mathrm{e}^{-t}\sqrt{1-(t/2\beta)}$$

$$Z(\beta, 2, t_{max}) = (2/\beta)^{3/2}\frac{1}{\pi}\int_0^{t_{max}} dt t^{1/2}\mathrm{e}^{-t}\sqrt{1-(t/2\beta)}$$

$$Z(\beta, 2, t_{max}) = (\beta\pi)^{-3/2}2^{1/2}\sum_{l=0}^{\infty} A_l(t_{max})(2\beta)^{-l} \ ,$$

with

$$A_l(t_{max}) \equiv \frac{\Gamma(l+1/2)}{l!(1/2-l)}\int_0^{t_{max}} dt \mathrm{e}^{-t} t^{l+1/2} \ ,$$

$$Z(\beta, t_{max}) = (\beta\pi)^{-3/2} 2^{1/2} \sum_{l=0}^{\infty} A_l(t_{max}) (2\beta)^{-l} \ ,$$

$$A_l(t_{max}) \equiv \frac{\Gamma(l+1/2)}{l!(1/2-l)} \int_0^{t_{max}} dt \mathrm{e}^{-t} t^{l+1/2} \ ,$$

When $t_{max} \to \infty$ the integral becomes the (complete) Γ function and the coefficients grow factorially. In lattice perturbation theory, we "add the tails" (to make the calculation easier).

When t_{max} is finite, the integral is bounded by a power of t_{max}. When $t_{max} \leq 2\beta$, the sum converges.

The peak of the integrand $\mathrm{e}^{-t} t^{k+1/2}$ becomes larger than 2β when $k \simeq 2\beta - 1/2$. The exact expansion is approximately truncated at that order.

Optimal order of truncation: $k^\star = 2\beta - 1/2$.

This is the same order as the order where the peak of the integrand moves outside of the range of integration in the exact expansion ($t_{max} = 2\beta$ and we have β dependent coefficients, instead of $t_{max} = \infty$ for the β-independent coefficients of the regular perturbative expansion; we can also pick optimal t_{max} at fixed order)

Using regular perturbation theory, we cannot do better than

$$Min_k \ |\Delta_k| \simeq (2^{1/2}/\pi)\beta^{-2}\mathrm{e}^{-2\beta} \ . \qquad (1)$$

This is the "perturbative envelope" of the "accuracy curves" at various orders

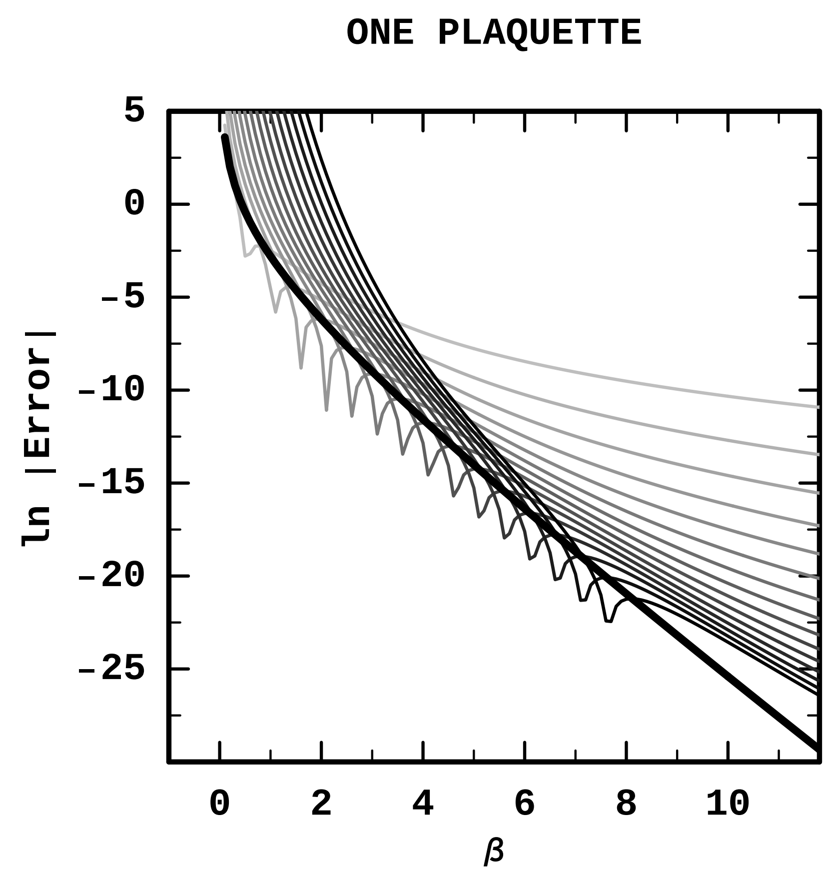

Figure 4: Natural logarithm of the absolute value of the difference between the series and the numerical value for order 1 to 10 for the one plaquette integral as a function of β. As the order increases, the curves get darker.The thicker dark curve is Ln $((2^{1/2}/\pi)\mathrm{e}^{-2\beta}/\beta^2)$.

Optimized cut perturbation theory

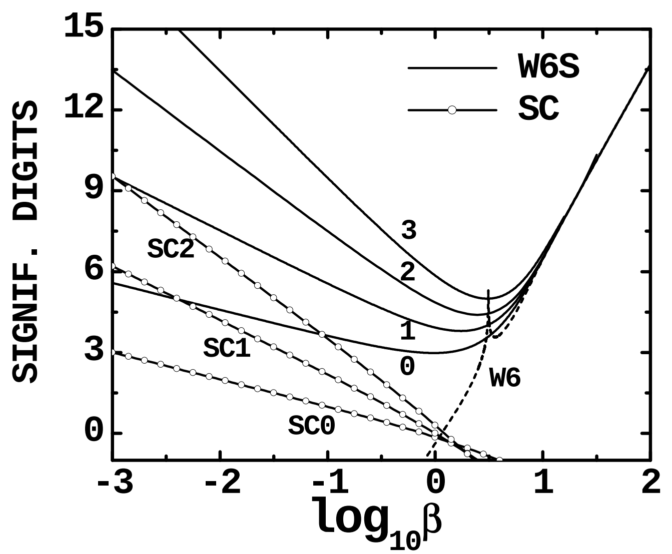

Figure 5: Significant digits obtained from the weak series truncated at order 6 using the first solution for t_{max}/β at order 0 to 3 compared to the weak coupling expansion at order 6 (dotted line W6) and the strong coupling expansion at order 0 to 2 (empty circles SC)

Separating the Perturbative and Non-Perturbative Parts

$$Z(\beta) = Z_{Pert.}(\beta) + Z_{NPert.}(\beta) \; , \tag{2}$$

$$Z_{Pert.}(\beta) = (\beta\pi)^{-3/2} 2^{1/2} \sum_{k=0}^{r(k^\star)} A_k(\infty)\beta^{-k} \; , \tag{3}$$

$$Z_{NPert.}(\beta) = (R(\beta) - T(\beta)) \tag{4}$$

$$R(\beta) = (\beta\pi)^{-3/2} 2^{1/2} \sum_{r(k^\star)+1}^{\infty} A_k(2\beta)\beta^{-k} \tag{5}$$

$$T(\beta) = (\beta\pi)^{-3/2} 2^{1/2} \times \sum_{k=0}^{r(k^\star)} \beta^{-k} \frac{\Gamma(k+1/2)}{k!(1/2-k)} \int_{2\beta}^{\infty} dt \mathrm{e}^{-t} t^{k+1/2} \; , \tag{6}$$

A surprising cancellation

A detailed calculation shows that at leading order, the removal of the added tails of integration cancels the remaining orders.

$$T(\beta) \simeq R(\beta) \simeq (2/\pi)^{3/2}\beta^{-3/2}\mathrm{e}^{-2\beta} \tag{7}$$

The two leading order contributions cancel and

$$Min_k \; |\Delta_k| \simeq (2^{1/2}/\pi)\beta^{-2}\mathrm{e}^{-2\beta} \; . \tag{8}$$

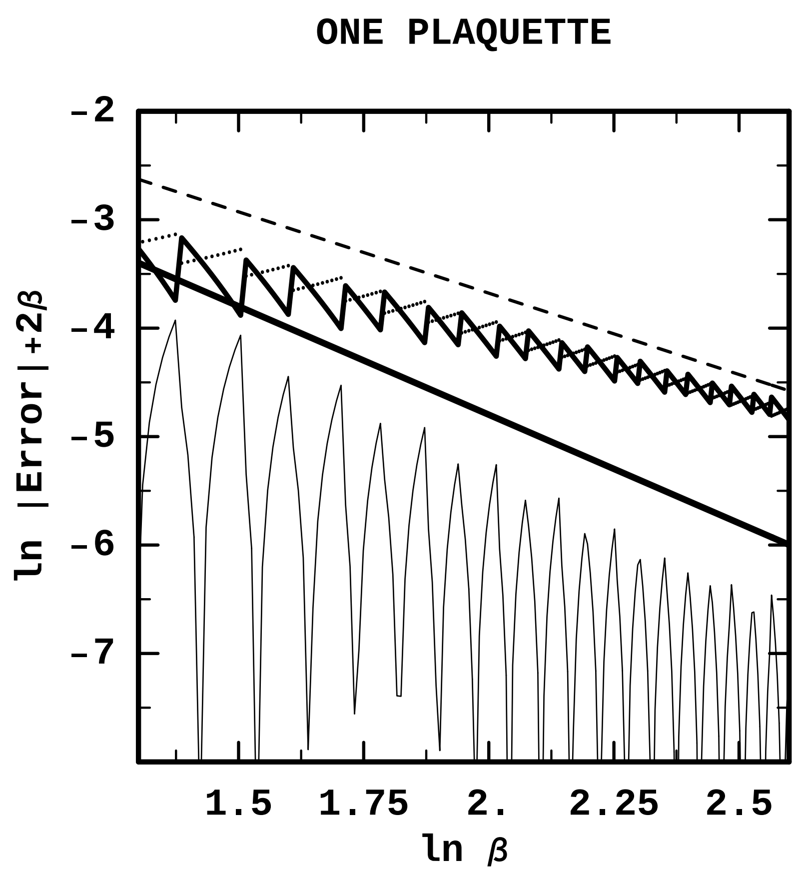

Figure 6: ln $|R(\beta) - T(\beta)| + 2\beta$ versus ln(β) (wavy curve in the lower part of the graph). The thicker line is ln $((2^{1/2}/\pi)/\beta^2)$. The dashed line is ln$((2/\pi)^{3/2}\beta^{-3/2})$. ln $|R(\beta)| + 2\beta$ (points) and ln $|T(\beta)| + 2\beta$ (line) are inter weaved between the two straight lines.

Quenched lattice QCD

$$S = \beta \sum_{plaq.} (1 - (1/N) ReTr(U_p)) \ , \tag{9}$$

with $\beta = 2N/g^2$. The lattice functional integral or partition function is

$$Z = \prod_l \int dU_l \mathrm{e}^{-S} \tag{10}$$

with dU_l the $SU(N)$ invariant Haar measure for the group element associated with the link l.

$< \mathcal{O} >$ is defined by inserting $\mathcal{O}$ in the integral and dividing by Z. We consider symmetric (hypercubic) lattices with L^D sites and periodic boundary conditions.

The total number of 1×1 plaquettes is denoted

$$\mathcal{N}_p \equiv \; L^D D(D-1)/2 \; . \tag{11}$$

Using

$$f \equiv -(1/\mathcal{N}_p) \ln Z \; , \tag{12}$$

we define the average plaquette

$$\begin{aligned} P(\beta) &\equiv \partial f/\partial \beta \\ &= (1/\mathcal{N}_p) \left\langle \sum_p (1 - (1/N) ReTr(U_p)) \right\rangle \; . \end{aligned} \tag{13}$$

Lattice Perturbation Theory

Three steps (Heller and Karsch, NPB 251 254)

1. $\beta = 2N/g^2$; $U = e^{igA}$ with $A = A^a T^a$

2. Extend the range of integration of the A^a from $-\infty$ to $+\infty$

3. Expand in g

We used the series of Di Renzo et al. JHEP 10 038 hep-lat/0011067.

$$P(1/\beta) = \sum_{m=0}^{10} b_m \beta^{-m} + \ldots.$$

Series analysis (L. Li and Y. M. PRD D73 036006)

- $P \propto (1/5.74 - 1/\beta)^{1.08}$
- not expected: zero radius of convergence for β^{-1} expansion (the plaquette changes discontinuously at $\beta \rightarrow \pm\infty$ (Li, YM PRD 71 016008 2005)
- good agreement with Horsley et al. hep-lat/0110210
- Not seen in 2d derivative of P (would requires massless glueballs!)

A Small Window for Complex Singularities

A simple alternative: the critical point in the fundamental-adjoint plane has mean field exponents and in particular $\alpha = 0$. On the $\beta_{adj.} = 0$ line, we assume an approximate logarithmic behavior (mean field)

$$-\partial P/\partial\beta \propto \ln((1/\beta_m - 1/\beta)^2 + \Gamma^2) \ , \tag{14}$$

This implies the approximate form

$$\partial^2 P/\partial\beta^2 \simeq -C \frac{(1/\beta_m - 1/\beta)}{\beta^3((1/\beta_m - 1/\beta)^2 + \Gamma^2)} \tag{15}$$

Typical Fits: $\beta_m \simeq 5.78$, $\Gamma \simeq 0.006$, and $C \simeq 0.15$

The stability of C and β_m can be used to set a lower bound on Γ. Given that the approximate form of $\partial^2 P/\partial\beta^2$ in Eq. (15) has extrema at $1/\beta = 1/\beta_m \pm \Gamma$. As we do not observe values larger than 0.3 near $\beta = 5.75$ we get the approximate bound $\frac{C}{2\beta_m^3\Gamma} < 0.3$ Large values of Γ would affect the low order coefficients. We never found estimate close to 0.01.

$$0.001 < \Gamma < 0.01 \ . \tag{16}$$

This suggests zeroes of the partition function in the complex β plane with

$$0.03 \simeq 0.001\beta_m^2 < Im\beta < 0.01\beta_m^2 \simeq 0.33 \tag{17}$$

No contradictions so far (work done with A. Denbleyker)

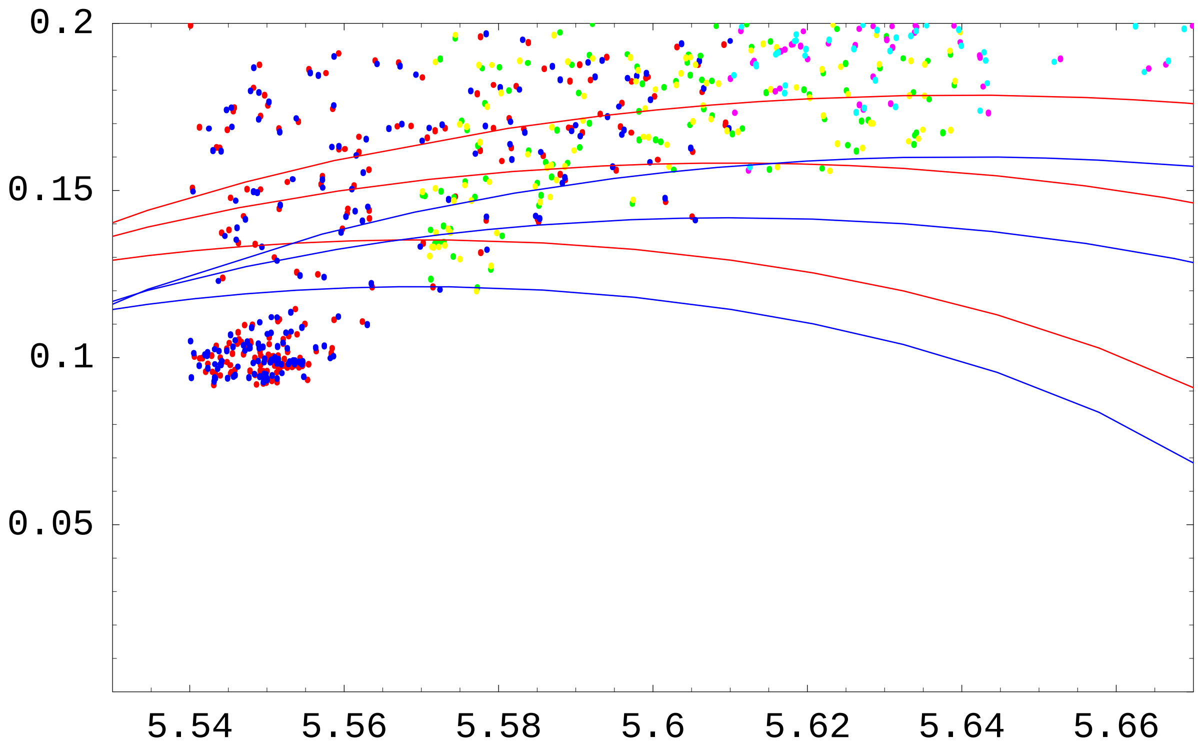

Figure 7: Zeroes of the partition function in the complex β plane for a 4^4 lattice (reproduces Alves and Berg). The dots correspond to disctinct bootstraps and the solid lines to the radius of confidence (work done with Alan denBleyker).

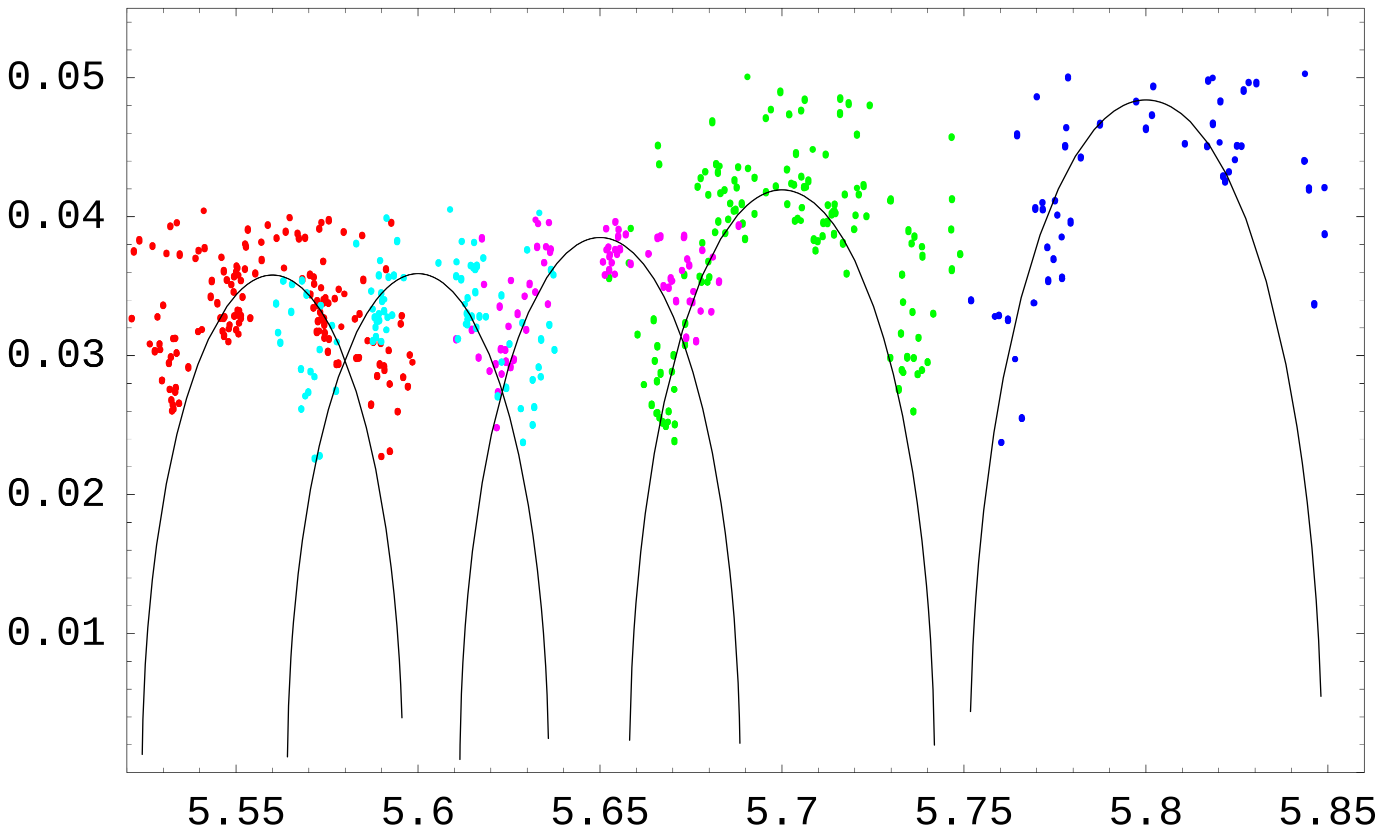

Figure 8: Zeroes of the partition function in the complex β plane for a 8^4 lattice. The dots correspond to disctinct bootstraps and the solid lines to the radius of confidence (work done with Alan denBleyker).)

Perturbative Envelope for Quenched QCD

A simple guess is that the envelope of the accuracy curves is given by a power of some renormalization group invariant scale. For instance, the fourth power of the two-loop renormalization group invariant scale,

$$Min_k\ |\Delta_k| \simeq C(\beta)^{204/121} \mathrm{e}^{-(16\pi^2/33)\beta}\ . \tag{18}$$

Fig. 9 shows that this provides a reasonable envelope in the region $5.5 < \beta < 6$ for $C \simeq 6.5 \times 10^8$. As β increases, the curves "leave" the conjectured envelope, they become more flat. For instance at order 8, for $6 < \beta < 7$, reasonable fits can be obtained (Burgio et al. 97) with Λ^2_{2loop}.

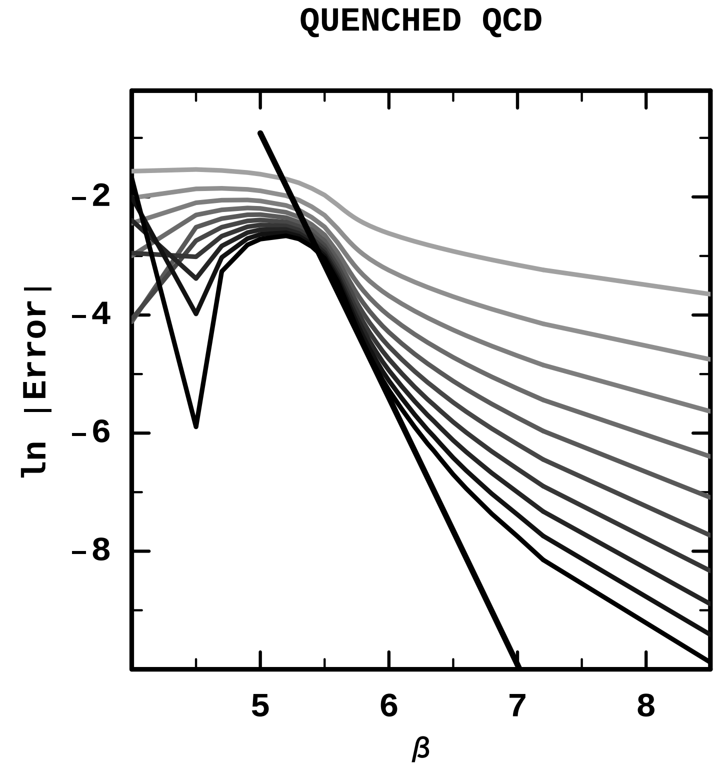

Figure 9: Natural logarithm of the absolute value of the difference between the series and the numerical value for order 1 to 10 for quenched QCD as a function of β. As the order increases, the curves get darker.The thicker dark curve is ln $(6.5 \times 10^8 \times (\beta)^{204/121} \mathrm{e}^{-(16\pi^2/33)\beta})$

Another simple guess is that the envelope is proportional to a power of the lattice spacing a expressed in units of $r_0 = 0.5$ fermi. For the interval $5.7 < \beta < 6.92$, the following power series is available (Necco and Sommer 01)

$$\ln(a/r_0) = -1.6804 - 1.7331\,(\beta - 6) + 0.7849\,(\beta - 6)^2 - 0.4428\,(\beta - 6)^3\,.$$

It has been suggested that the envelope is proportional to a^4 (Horsley 01, Rakow 05)

This is hard to establish without the knowledge of the higher orders.

On the other hand, the accuracy curve at order 10 can be fitted very well with a^2.

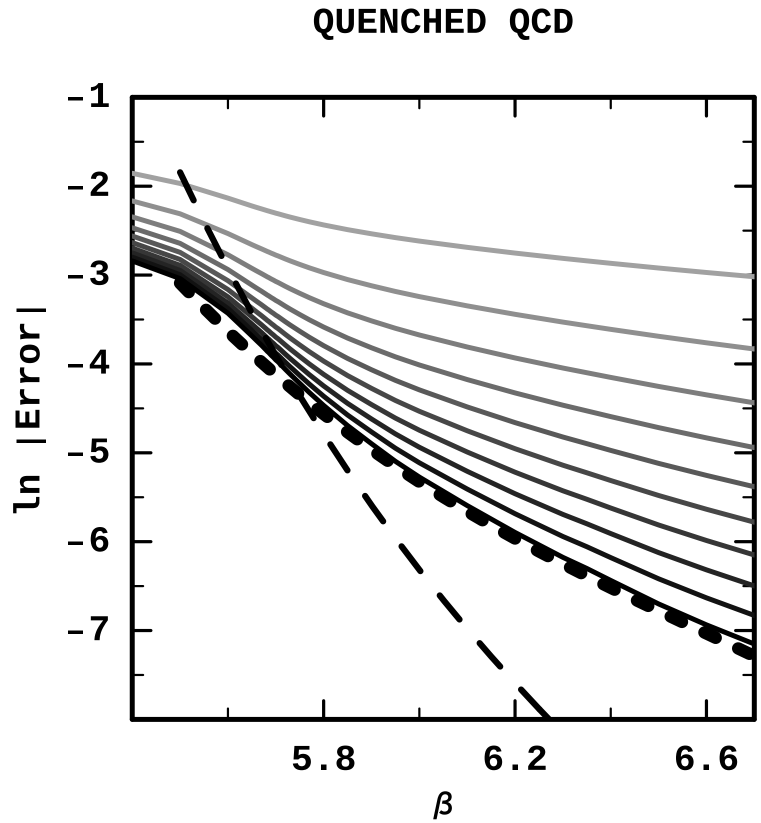

Figure 10: Natural logarithm of the absolute value of the difference between the series and the numerical value for order 1 to 10 for quenched QCD as a function of β. As the order increases, the curves get darker. The short dash curve is ln$(0.14\ (a/r_0)^2)$, The long dash curve is ln$(1.5\ (a/r_0)^4)$.

The empirical error is larger than the next order contribution.

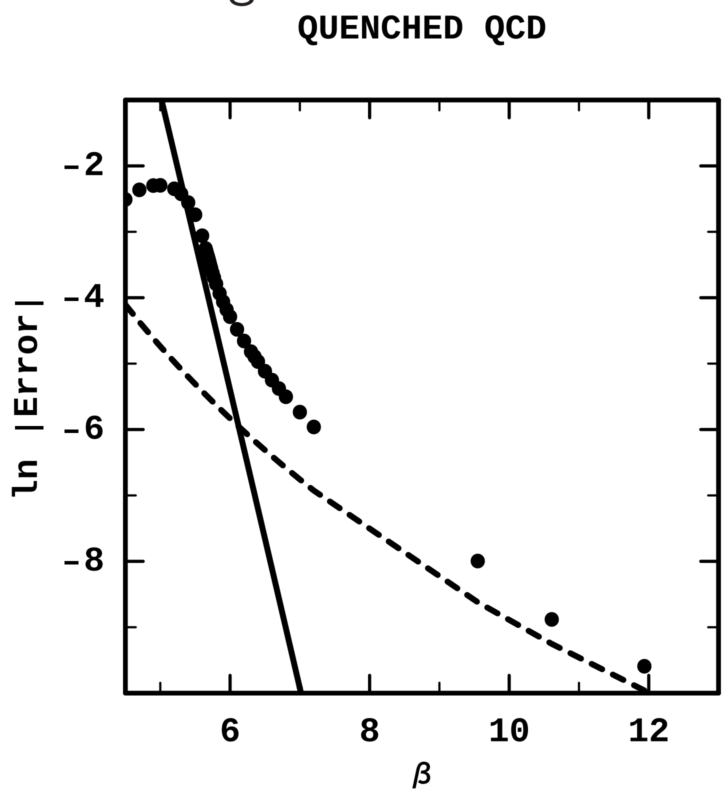

Figure 11: Natural logarithm of the absolute value of the difference between the series and the numerical value for order 5 for quenched QCD as a function of β (dots). The dash line is $\ln(|a_6/\beta^6|)$. The solid curve is ln $(6.5 \times 10^8 \times (\beta)^{204/121} \mathrm{e}^{-(16\pi^2/33)\beta})$.

Large order extrapolations

Model 1:

$\sum_{k=0} b_k \beta^{-k} \simeq C(\mathrm{Li}_2(\beta^{-1}/(\beta_m^{-1} + i\Gamma)) + \mathrm{h.c},$

$\mathrm{Li}_2(x) = \sum_{k=0} x^k/k^2$.

We fixed $\Gamma = 0.003$ and obtained $C = 0.0654$ and β_m=5.787 using of a_9 and a_{10}. The low order coefficients depend very little on Γ

It works very well!

order	predicted	calculated
1	0.7567	2
2	1.094	1.2208
3	2.811	2.961
4	9.138	9.417
5	33.79	34.39
6	135.5	136.8
7	575.1	577.4
8	2541	2545
9	11590	11590
10	54160	54160

Also $a_{16} = 7.7\ 10^8$ while from Fig. 1 of P. Rakow Lattice 2006 $a_{16} = 0.00027 \times 6^{16} = 7.6\ 10^8$;

Feynman diagram interpretation ???

Model 2 (Mueller 93, di Renzo 95):

$$\sum_{k=0} b_k \bar{\beta}^{-k} \simeq K \int_{t_1}^{t_2} dt \mathrm{e}^{-\bar{\beta}t} \, (1 - t \; 33/16\pi^2)^{-1-204/121} \tag{19}$$

$$\bar{\beta} = \beta(1 + d_1/\beta + \dots) \tag{20}$$

$t_1 = 0$ corresponds to the UV cutoff

$t_2 = 16\pi^2/33$: Landau pole.

$t_2 = \infty$: usual perturbative series

$$Min_k\ |\Delta_k| \simeq 3.5(\bar{\beta})^{204/121-1/2}\mathrm{e}^{-(16\pi^2/33)\bar{\beta}}\ . \tag{21}$$

Except for the 1/2 in the exponent this the two loop invariant.

Shifting to β using Eq. (20) and neglecting β^{-1} corrections

$$Min_k\ |\Delta_k| \simeq 3.1\ 10^8(\beta)^{204/121-1/2}\mathrm{e}^{-(16\pi^2/33)\beta}\ . \tag{22}$$

The two extrapolations models are compared in Fig. 12. The two models yields similar coefficients up to order 20. After that, the integral model has the logarithm of its coefficients growing faster than linear.

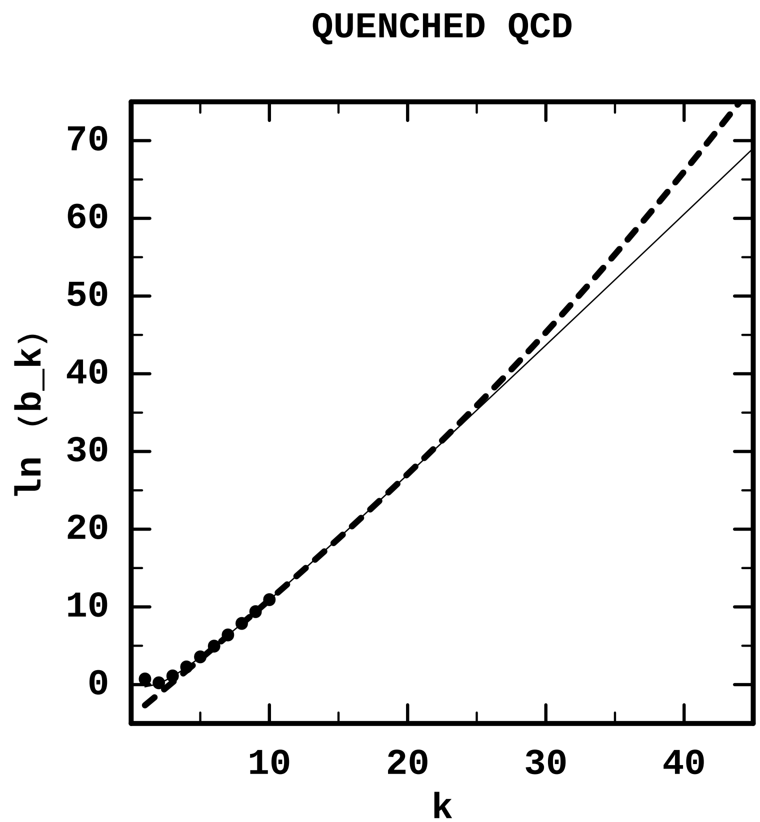

Figure 12: ln(b_k) for the dilogarithm model (solid line) and the integral model (dashes). The dots up to order 10 are the known values.

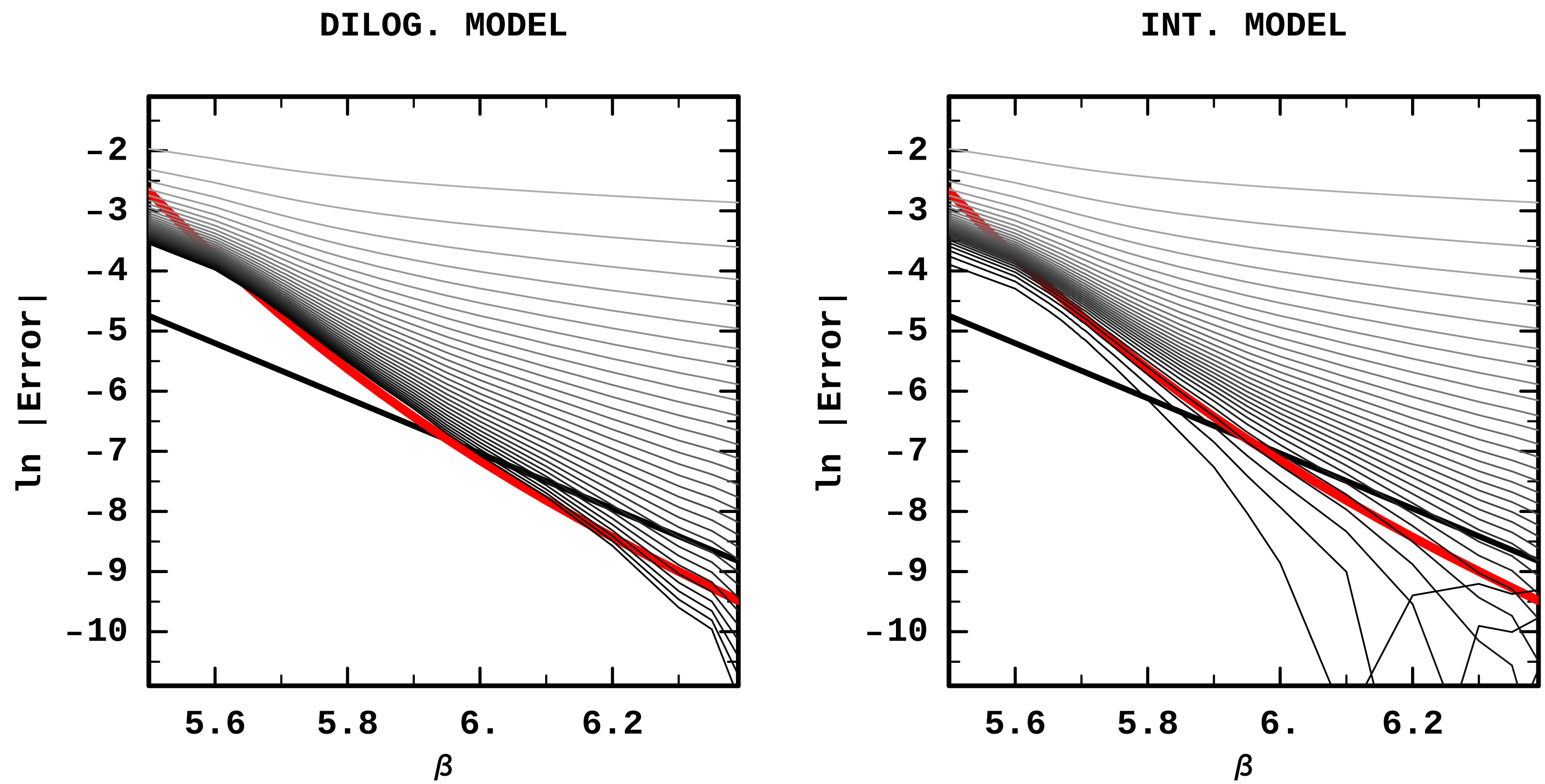

Figure 13: Accuracy curves for the dilogarithm model (left) and the integral model (right). As The long dash curve is ln$(0.65\ (a/r_0)^4)$. The solid curve is ln $(3.1 \times 10^8 \times (\beta)^{204/121-1/2} \mathrm{e}^{-(16\pi^2/33)\beta})$

Parametrization of the force scale

As a^4 behavior looks plausible, can we parametrize it with an exponential?

$$d\ln(a/r_0)/d\beta = -(4\pi^2/33) + (51/121)\beta^{-1} + A\mathrm{e}^{-B\beta} \quad (23)$$

with $A = -1.35\ 10^7$ and $B = 2.82$

$$\ln(a/r_0) = C - (4\pi^2/33)\beta + (51/121)\ln(\beta) - (A/B)\mathrm{e}^{-B\beta} \quad (24)$$

with $C = 4.5$

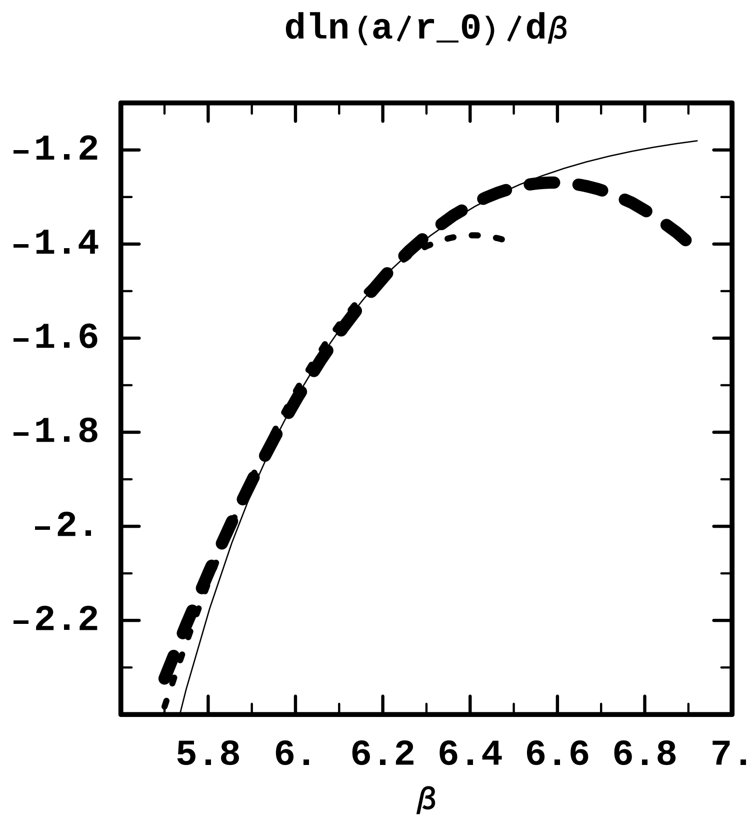

Figure 14: $d\ln(a/r_0)/d\beta$ using Necco 01 (thick dashes), Guagnelli 98 (small dashes) and our parametrization (solid line).

Summary of expectations for quenched QCD

- Series dominated by complex singularity up to order ≈ 20
- Rule of thumb can be applied after ($\beta = 6$: truncation at order 25)
- a^4 looks plausible, but constant of proportionality hard to get.
- Semi-classical program looks feasible

Work in progress

MC calculations of pert. coefficients (with M. Naides)

Stochastic quantization with a field cutoff

Dispersion relations (with D. Du)

Semi-classical calculations of the non-perturbative amplitudes

Scaling variable calculation of the perturbative amplitude.