Gross-Neveu model as a laboratory for fermion discretization

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- Models, scheme, simulation
- Example results, Wilson: $N = 2, 8$
- Staggered (no results yet), Outlook

The models

Dirac $\leftrightarrow$ Majorana fermions, free Euclidean theory:

$$\mathcal{L}_0 = \sum_{k=1}^{N/2} \bar{\psi}_k (\partial\!\!\!/ + m)\, \psi_k = \frac{1}{2} \sum_{k=1}^{N} \xi_k^\top \underbrace{\mathcal{C}(\partial\!\!\!/ + m)}_{\text{antisymmetric}} \xi_k$$

with

$$\begin{pmatrix} \psi_1 \\ \bar{\psi}_1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \xi_1 + i\xi_2 \\ (\xi_1^\top - i\xi_2^\top)\mathcal{C} \end{pmatrix}, \text{etc}, \quad \mathcal{C}\gamma_\mu \mathcal{C}^{-1} = -\gamma_\mu^\top \, [\Rightarrow \mathcal{C}^\top = -\mathcal{C}]$$

$\Rightarrow$ large internal 'flavor' symmetry $\mathrm{O}(N) \supset \mathrm{U}(N/2)$

by insisting on O(N) invariance:

- only one 4-fermion interaction possible

$$\mathcal{L}_{\mathrm{I}} = -\frac{g^2}{8}(\xi_i^{\top}\mathcal{C}\xi_i)^2$$

- renormalizable in $D=2$
  - $N=2$: Thirring model
  - $N \geqslant 3$: original asymptotically free Gross-Neveu models
- also odd N possible
- discrete chiral invariance $\xi_i \to \gamma_5 \xi_i$ iff $m=0$ (Wilson: $\to m_c$)
- GN with chiral U(1) needs three couplings (mixing under renormalization!) $\Rightarrow$ disfavored for precision tests

Finite size renormalization scheme

- $T \times L$ torus, $\xi(x \pm T\hat{0}) = -\xi(x), \xi(x \pm L\hat{1}) = \xi(x)$

- IR-regularized by smallest momentum $p_* = (\pi/T, 0)$

- formulate nonperturbative chiral renormalization conditions at scale T

- momentum space: normalize at momentum $p_* \rightarrow$ not discussed here

- position space, $T = L$, PT refers to Wilson fermions $[r=1]$

$$\xi_R = Z_\xi^{-1/2}\,\xi, \qquad \breve{\xi}(x_0) = a\sum_{x_1}\xi(x)$$

$$\left\langle \breve{\xi}_1^\top(T/4)\mathcal{C}\breve{\xi}_1(0)\right\rangle = 0 \Rightarrow m = m_c\ (\rightarrow \text{tune})$$

PT: $\quad am_c = -(N-1)\,Kg^2 + \mathrm{O}(a^4g^2, g^4), \qquad K = 0.384900179460$

$$Z_\xi = -\frac{1}{L}\left\langle \breve{\xi}_1^\top(T/2)\mathcal{C}\gamma_0\breve{\xi}_1(0)\right\rangle\ [= 1 + \mathrm{O}(g^4)\ \text{in PT}]$$

$$g_R^2 = \frac{4}{TL}\left\langle \breve{\xi}_{R,1}^\top(T/2)\mathcal{C}\,\breve{\xi}_{R,1}(0)\ \breve{\xi}_{R,2}^\top(T/2)\mathcal{C}\,\breve{\xi}_{R,2}(0)\right\rangle$$

PT: $\quad g_R^2 = \frac{T}{T-2a}\left(g^2 + \left[\frac{N-2}{2\pi}\ln(L/a) + c_0 + \mathrm{O}(a)\right] g^4 + \mathrm{O}(g^6)\right)$

$$c_0 = 0.309652\ldots. - (N-2)\times 0.483524\ldots.$$

Simulation

$$\mathrm{e}^{\frac{g^2}{8}a^2\sum_x(\xi_i^\top \mathcal{C}\xi_i)^2} = \int \prod_x d\mu(\sigma)\,\mathrm{e}^{-\frac{g}{2}a^2\sum_x \sigma\xi_i^\top\mathcal{C}\xi_i}$$

valid if $\int d\mu(\sigma)\left\{\begin{array}{l}\sigma^{2n}\\ \sigma^{2n+1}\end{array}\right\} = \left\{\begin{array}{l}\frac{2n!}{n!}(2a^2)^{-n}\\ 0\end{array}\right\}$ for $n \leqslant N/2$

standard: $d\mu = \text{Gaussian}$ (for all N) $\Rightarrow$

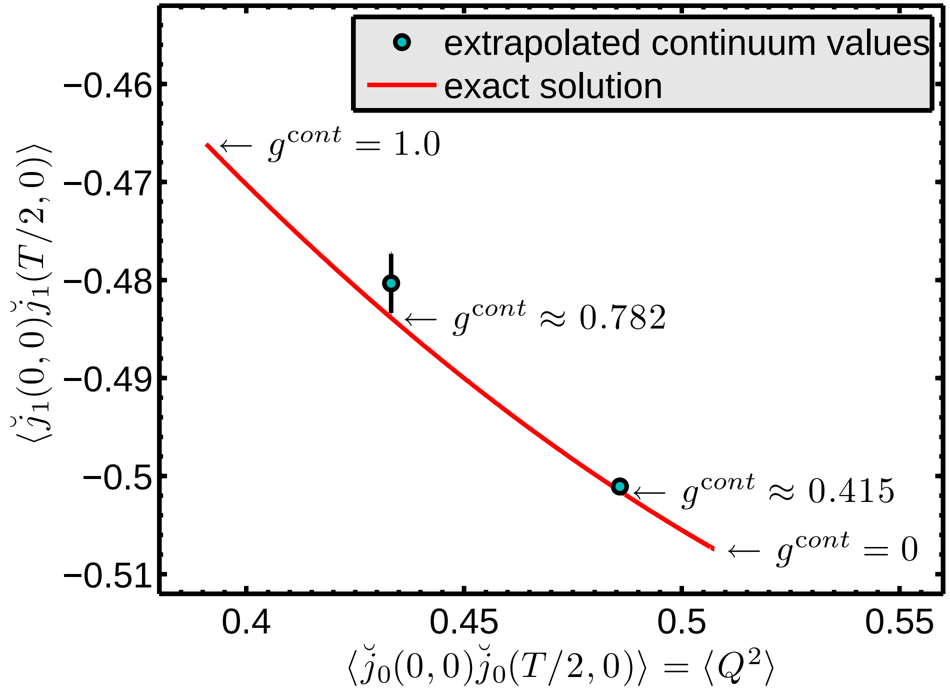
$$Z = \int \prod_x d\sigma\,\mathrm{e}^{-\frac{1}{2}a^2\sum_x\sigma^2}\underbrace{[\mathrm{Pf}\,A(\sigma)]^N}_{=(\det A)^{N/2}}$$

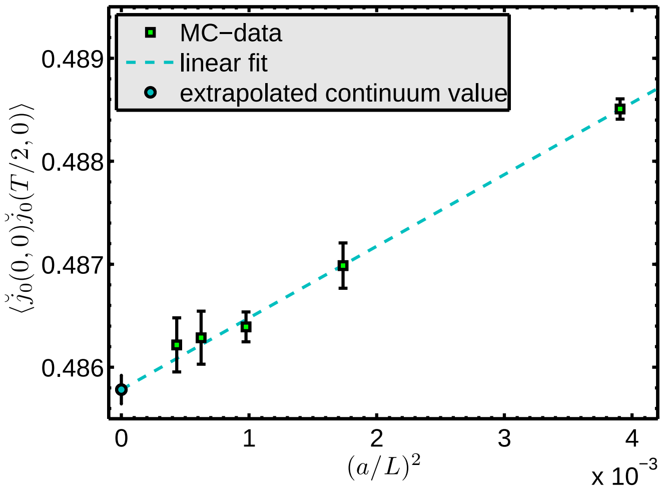
$$A(\sigma) = \mathcal{C}(\gamma_\mu\partial_\mu + m + g\sigma - a\partial^*\partial)$$

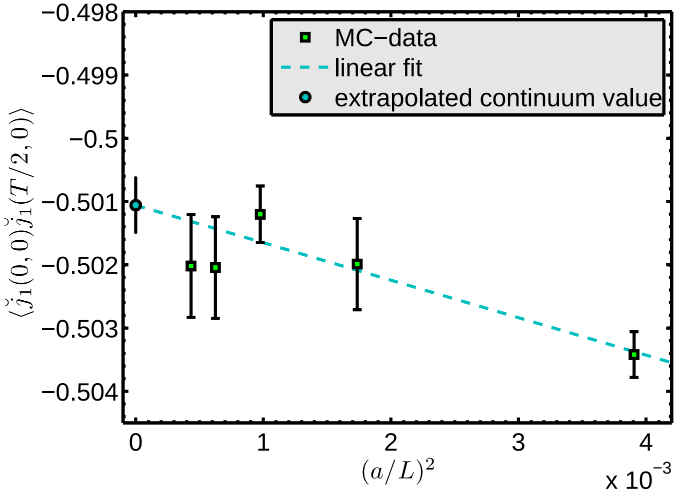
- A antisymmetric matrix, real with Majorana γ_μ
- for even $N \Rightarrow 0 \leqslant (\mathrm{Pf}\,A)^N = (\det A)^{N/2} = (\det A^\top A)^{N/4}$
- simulable by standard methods ($N/2$ real pseudofermions)

$N = 2$: Massless Thirring model

- exact solution in the continuum ($T = L = \infty$)
- only finite coupling renormalization, no running, $\beta \propto (N - 2)$
- key formula: $\det(\not\partial + i\not A) = \exp\left(-\frac{1}{2\pi} \int d^2x\, A_\mu [\delta_{\mu\nu} - \frac{\partial_\mu \partial_\nu}{\partial^2}] A_\nu \right)$
- extended to $T, L < \infty$ (complicated by zeromodes)
- correlations of $\breve{j}_\mu(x_0, p) = \int_0^L dx_1\, \mathrm{e}^{-ipx_1}\, \bar\psi \gamma_\mu \psi$ worked out
- $\breve{j}_0(x_0, 0) = Q$ charge,
- $\Rightarrow$ check with Wilson fermions
    - continuum limit at fixed bare $g = 0.4,\ 0.7$
      $L/a = 16 \ldots 48$ always tuning $m = m_c(g)$
    - use Noether current ($Z_V = 1$)


extrapolated continuum values
exact solution
$\leftarrow g^{cont} = 1.0$
$\leftarrow g^{cont} \approx 0.782$
$\leftarrow g^{cont} \approx 0.415$
$\leftarrow g^{cont} = 0$
−0.46
−0.47
−0.48
−0.49
−0.5
−0.51
0.4
0.45
0.5
0.55
$\langle \breve{j}_1(0,0)\breve{j}_1(T/2,0)\rangle$
$\langle \breve{j}_0(0,0)\breve{j}_0(T/2,0)\rangle = \langle Q^2\rangle$

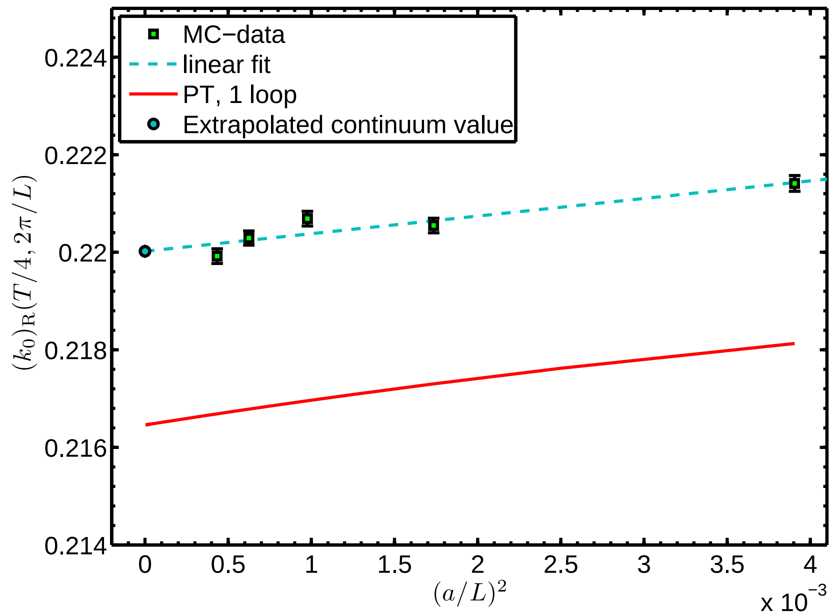

MC−data
linear fit
extrapolated continuum value
0.489
0.488
0.487
0.486
0
1
2
3
4
x 10^{-3}
$(a/L)^2$
$\langle \breve{j}_0(0,0)\breve{j}_0(T/2,0)\rangle$

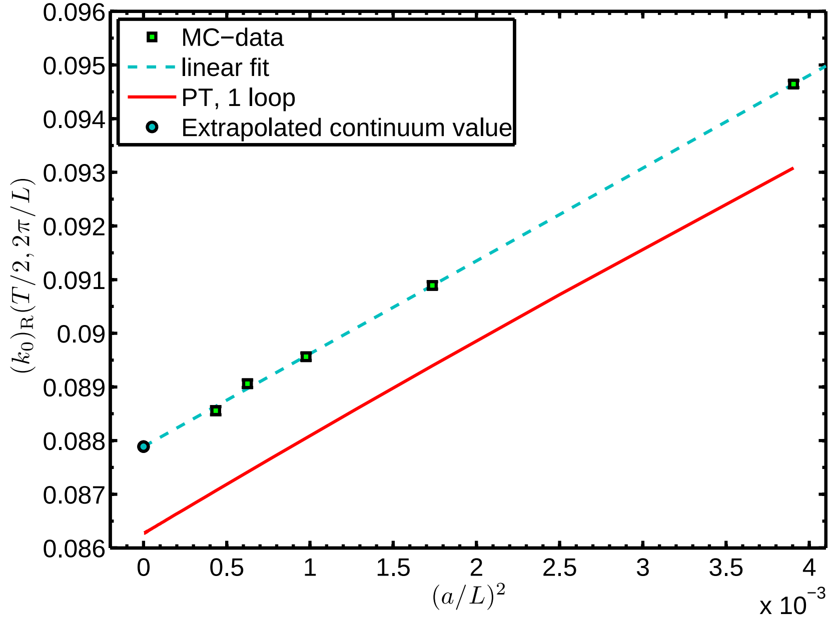

MC−data
linear fit
extrapolated continuum value
−0.498
−0.499
−0.5
−0.501
−0.502
−0.503
−0.504
0
1
2
3
4
x 10^{-3}
$(a/L)^2$
$\langle \breve{j}_1(0,0)\breve{j}_1(T/2,0)\rangle$

N=8 Gross Neveu

- many universal correlations definable
- one example (at $g_R = 0.38$)

$$(k_\mu)_R(x_0, p) = -\frac{1}{Z_\xi} a \sum_{x_1} \mathrm{e}^{-ipx_1} \langle \xi^\top(0) \mathcal{C} \gamma_\mu \xi(x) \rangle$$





Staggered Fermions

D dimensions

- naive fermions: 2^D 'tastes' (Dirac and Majorana)
- reduction factor by "spin diagonalization":
  Dirac: $2^{D/2}$, Majorana: $2^{D/2-1}$ (none in $D=2$)
- reason: $\mathcal{C}$ not transformed away (has 2×2 blocks)
- $D=2$: 2 Dirac- or 4 Majorana-tastes ($\times$ 2 to simulate)
- $m_c=0$, no additive renormalization
- naive 4-fermion term:

$$\frac{1}{(TL)^4}\sum_{p_1,\dots,p_4}\delta^2\Big(\sum p_i\Big)\underbrace{\tilde{\xi}_i^\top(p_1)\mathcal{C}\,\tilde{\xi}_i(p_2)}_{\text{taste}-\text{mixing!}}\,\tilde{\xi}_j^\top(p_3)\mathcal{C}\,\tilde{\xi}_j(p_4)$$

$\rightarrow$ extra factor: $\prod_\mu \cos(a\,(p_1+p_2)_\mu/2)\cos(a\,(p_3+p_4)_\mu/2)$

locally square in position space:

$$\frac{1}{8}\Big[\xi^\top \mathcal{C}\xi\,(x) + \xi^\top \mathcal{C}\xi\,(x+a\hat{0}) + \xi^\top \mathcal{C}\xi\,(x+a\hat{1}) + \xi^\top \mathcal{C}\xi\,(x+a\hat{0}+a\hat{1})\Big]$$

- O($4N$) Gross-Neveu in the continuum limit??
- compare universal correlations

Summary, Outlook

- Finite size scheme for GN
- relatively easy to tune chiral (similar to Schrödinger functional in QCD), also for Wilson, + continuum limit
- easy to define universal correlations (ratios...)
- precise universality check, staggered, Wilson, ... possible?
- Nonstandard fermion algorithms? Dimer? World line? Other factorizations of $(\xi^\top \mathcal{C}\xi)^2$?