Mass Splitting of Staggered Fermion and $SO(2D)$ Clifford Algebra

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   $\Rightarrow$ **K. Itoh , M. Kato , M. Murata , H. Sawanaka , H. So (IKMSS, 2005)**

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1. Motivation

Staggered Fermion

The species doubling $\Longrightarrow$ Staggered Fermion

Doubling $\longrightarrow$ Spinor + *Taste*

Theoretical issues

The 4-fold degeneracy problem $\Longrightarrow$ Fourth-Root Trick

$$detD = (detD_{st})^{1/4} \cdots$$

(*Cf* : Y. Shamir , Nucl. Phys. B153 (2006) 291)

What's new in the present work

♠ Reformulation of Staggered Fermion ♠

♠ New four operators which keep the rotational invariance ♠

To obtain a light taste mode $\cdots$

$$\bar{\Psi} D^{Imp} \Psi \equiv \bar{\Psi} D_{st} \Psi + \text{New operators}$$

(2-dim. and $U = 1$ in this study)

(*Cf* : P. Mitra and P. Weisz , Phys. Lett. 126B (1983) 355)

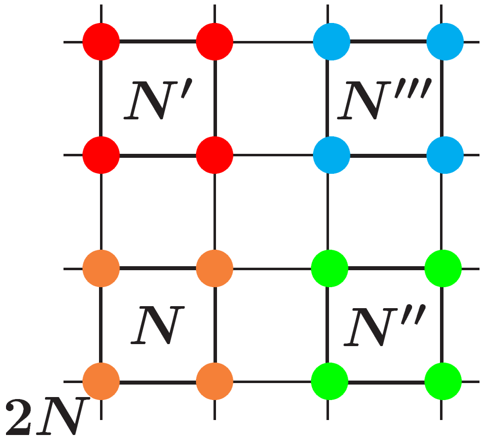
2. Reformulation of Staggered Fermion

IKMSS , PTP 114 (2005) 631

- Reformulation on D-dimensional lattice space

based on $SO(2D)$ Clifford algebra

$$\Psi(n) \equiv \Psi_r(N) : \quad \begin{array}{ccccc} Doubling & \longrightarrow & Spinor & + & Taste \\ 2^D & \longrightarrow & 2^{D/2} & \times & 2^{D/2} \\ SO(2D) & \longrightarrow & SO(D) & \times & SO(D) \end{array}$$



$n_\mu \equiv 2N_\mu + r_\mu$

N_μ : some integer

r_μ : 0 or 1 components

$\Psi_r(N)$: $SO(2D)$ spinor

- Staggered fermion action

$$S = \bar{\Psi}_r(N)(D_{st})_{(r,N),(r',N')}\Psi_{r'}(N')$$

$$= \bar{\Psi}_r(N)\left[\sum_{\mu,\vec{\varepsilon}}(\Gamma_{\mu,\vec{\varepsilon}})_{(r,r')}(D_{\mu,\vec{\varepsilon}})_{(N,N')}\right]\Psi_{r'}(N')$$

- $(D_{\mu,\vec{\varepsilon}})_{(N,N')}$: Generalized difference operator

$$(D_{\mu,\vec{\varepsilon}})_{(N,N')} = \frac{1}{2^D}\sum_{\vec{\sigma}}(-1)^{\vec{\varepsilon}\cdot\vec{\sigma}}(\nabla_\mu^{\vec{\sigma}})_{(N,N')} \ ,$$

$$(\nabla_\mu^{\vec{\sigma}})_{(N,N')} = \begin{cases} \frac{1}{2a}(\delta_{N,N'}U_{2N+\vec{\sigma},\mu} - \delta_{N-\hat{\mu},N'}U^\dagger_{2N+\vec{\sigma}-\hat{\mu},\mu}), & \sigma_\mu = 0, \\ \frac{1}{2a}(\delta_{N+\hat{\mu},N'}U_{2N+\vec{\sigma},\mu} - \delta_{N,N'}U^\dagger_{2N+\vec{\sigma}-\hat{\mu},\mu}), & \sigma_\mu = 1, \end{cases}$$

$\vec{\varepsilon}$, $\vec{\sigma}$: 2-dimensional vector with its component of 0 or 1

$U_{2N+\vec{\sigma},\mu}$: link variable (In this study, $U = 1$)

- $(\Gamma_{\mu,\vec{\varepsilon}})_{(r,r')}$:

Basis of $SO(2D)$ Clifford algebra $\cdots$ $\Gamma_{\mu,\vec{0}} = \gamma_\mu$, $\Gamma_{\mu,\vec{e}_\mu} = i\tilde{\gamma}_\mu$

$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_{\mu\nu}, \quad \{\gamma_\mu, \tilde{\gamma}_\nu\} = 0.$$

$\vec{e}_\mu$: unit vector along the μ-direction

In terms of γ and $\tilde{\gamma}$ $\cdots$ $\Gamma_{\mu,\vec{\varepsilon}} \equiv (i\tilde{\gamma}_1\gamma_1)^{\varepsilon_1}\cdots(i\tilde{\gamma}_D\gamma_D)^{\varepsilon_D}\gamma_\mu$

$\pi/2$ Rotational Symmetry

IKMSS , PTP 114 (2005) 631

For $\pi/2$ rotational transformation ($\mu(<)\nu$ plane),

$$N \to R(N), \quad r \to R(r), \quad \Psi(N) \to V_{\mu\nu}\Psi(R(N)), \quad D_{st} \to V_{\mu\nu} D_{st} V_{\mu\nu}^{\dagger}.$$

- $V_{\mu\nu}$: rotation matrix for spinor index, r

$$V_{\mu\nu} = \frac{e^{i\vartheta}}{2}\Gamma_{2D+1}(\tilde{\gamma}_\mu - \tilde{\gamma}_\nu)(1 + \gamma_\mu\gamma_\nu) \quad , \quad \Gamma_{2D+1} \sim \gamma_1 \cdots \gamma_D \tilde{\gamma}_1 \cdots \tilde{\gamma}_D$$

$$\{\Gamma_{2D+1}, \gamma_\mu\} = \{\Gamma_{2D+1}, \tilde{\gamma}_\mu\} = \{\Gamma_{2D+1}, \Gamma_{\mu,\vec{\varepsilon}}\} = 0$$

$e^{i\vartheta}$: Phase factor

The transformations of the basis of the $SO(2D)$ Clifford algebra :

$$V_{\mu\nu}\gamma_\rho V_{\mu\nu}^{\dagger} = \begin{cases} \gamma_\rho & (\rho \neq \mu, \nu) \\ -\gamma_\nu & (\rho = \mu) \\ \gamma_\mu & (\rho = \nu) \end{cases} \quad , \quad V_{\mu\nu}\tilde{\gamma}_\rho V_{\mu\nu}^{\dagger} = \begin{cases} \tilde{\gamma}_\rho & (\rho \neq \mu, \nu) \\ +\tilde{\gamma}_\nu & (\rho = \mu) \\ \tilde{\gamma}_\mu & (\rho = \nu) \end{cases}$$

- Rotationally invariant 4-operators : $\bar{\Psi}\mathcal{O}\Psi$

$$\begin{aligned} \mathcal{O}_1 &= 1 \\ \mathcal{O}_2 &= \Gamma_{D+1} \sim \gamma_1 \cdots \gamma_D \\ \mathcal{O}_3 &= \tilde{\gamma}_1 + \tilde{\gamma}_2 + \cdots + \tilde{\gamma}_D \\ \mathcal{O}_4 &= \Gamma_{D+1}(\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_D) \sim \gamma_1 \cdots \gamma_D(\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_D) \end{aligned}$$

$$\Psi(N) \to V_{\mu\nu}\Psi(R(N)), \quad D_{st} \to V_{\mu\nu} D_{st} V^{\dagger}_{\mu\nu}.$$

$$\begin{aligned}
\mathcal{O}_1 &= 1 \\
\mathcal{O}_2 &= \Gamma_{D+1} \sim \gamma_1 \cdots \gamma_D \\
\mathcal{O}_3 &= \tilde{\gamma}_1 + \tilde{\gamma}_2 + \cdots + \tilde{\gamma}_D \\
\mathcal{O}_4 &= \Gamma_{D+1}(\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_D) \sim \gamma_1 \cdots \gamma_D (\tilde{\gamma}_1 + \cdots + \tilde{\gamma}_D)
\end{aligned}$$

$\bar{\Psi} D_{st} \Psi$ and $\bar{\Psi}\mathcal{O}\Psi$ are invariant under the rotation.

- Improved Staggerd fermion action (keep the rotational invariance)

$$\begin{aligned}
S^{Imp} &\equiv \bar{\Psi} D^{Imp} \Psi \\
&\equiv \bar{\Psi} D_{st} \Psi + \bar{\Psi} M_R \Psi \\
&\equiv \bar{\Psi} \Gamma_{\mu,\vec{\varepsilon}} D_{\mu,\vec{\varepsilon}} \Psi + \bar{\Psi}(m_1 \mathcal{O}_1 + m_2 \mathcal{O}_2 + m_3 \mathcal{O}_3 + m_4 \mathcal{O}_4)\Psi
\end{aligned}$$

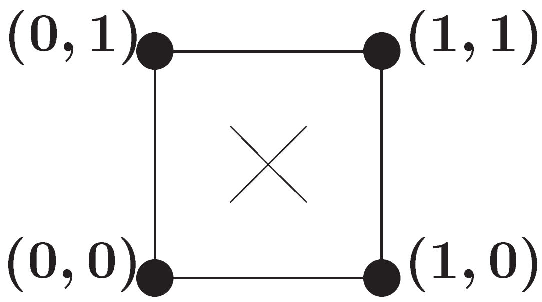
m_1, m_2, m_3, m_4 : Parameter

$\Longrightarrow$ Analysis of M_R ··· Section 3.1

Analysis of D^{Imp} ··· Section 3.3

$SO(4)$ Clifford algebra

$(r_1, r_2) =$



$SO(4)$ spinor

$$\Psi_r^T = (\Psi_{(0,0)}, \Psi_{(0,1)}, \Psi_{(1,0)}, \Psi_{(1,1)})$$

$$\gamma_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \gamma_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \Gamma_3 = i\gamma_1\gamma_2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{\gamma}_1 = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \tilde{\gamma}_2 = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}, \Gamma_5 = \gamma_1\gamma_2\tilde{\gamma}_1\tilde{\gamma}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$V_{12}(e^{i\vartheta} = 1) = \begin{pmatrix} 0 & 0 & -i & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & i & 0 & 0 \end{pmatrix}, V_{21}(e^{i\vartheta} = 1) = \begin{pmatrix} 0 & -i & 0 & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & 0 & -i & 0 \end{pmatrix}.$$

3. Mass splitting of Staggered Fermion

3.1 Mass matrix analysis

- General form of rotational invariant mass matrix

$$
\begin{aligned}
M_R &= m_1\mathcal{O}_1 + m_2\mathcal{O}_2 + m_3\mathcal{O}_3 + m_4\mathcal{O}_4 \\
&= m_1 1 + m_2\Gamma_3 + m_3(\tilde{\gamma}_1 + \tilde{\gamma}_2) + m_4\Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2) \\
&= \begin{pmatrix} m_1 & -im_3 + m_4 & -im_3 - m_4 & -im_2 \\ im_3 + m_4 & m_1 & -im_2 & -im_3 + m_4 \\ im_3 - m_4 & im_2 & m_1 & im_3 + m_4 \\ im_2 & im_3 + m_4 & -im_3 + m_4 & m_1 \end{pmatrix}
\end{aligned}
$$

m_1, m_2, m_3, m_4 : Parameter

- Eigenvalues for M_R

$$
\begin{cases} m_1 - m_2 - \sqrt{2}m_3 + \sqrt{2}m_4 \\ m_1 - m_2 + \sqrt{2}m_3 - \sqrt{2}m_4 \\ m_1 + m_2 - \sqrt{2}m_3 - \sqrt{2}m_4 \\ m_1 + m_2 + \sqrt{2}m_3 + \sqrt{2}m_4. \end{cases}
$$

Mass splitting $\cdots 2^2 \to 1 + 1 + 1 + 1$

$$\text{Eigenvalues for } M_R = \begin{cases} m_1 - m_2 - \sqrt{2}m_3 + \sqrt{2}m_4 \\ m_1 - m_2 + \sqrt{2}m_3 - \sqrt{2}m_4 \\ m_1 + m_2 - \sqrt{2}m_3 - \sqrt{2}m_4 \\ m_1 + m_2 + \sqrt{2}m_3 + \sqrt{2}m_4. \end{cases}$$

- **2-dimensional Dirac spinor is composed of a 2-component.**

$\Longrightarrow$ By tuning parameters,

it is possible to obtain a light Dirac mode, **2** :

$$\begin{cases} 2^2 \longrightarrow \mathbf{2} + \underline{1+1} \quad (e.g. \quad m_3 = m_4) \\ \qquad\qquad \longrightarrow \text{Heavy mode} \neq \text{Dirac spinor} \\ 2^2 \longrightarrow \mathbf{2} + 2 \end{cases}$$

- $2^2 \longrightarrow 2 + 2 \cdots$ 3-cases

	Mass eigenvalues	Parameter condition
Case 1	$m_1 \pm \sqrt{2}m_4$	$m_2 = m_3 = 0$
Case 2	$m_1 \pm \sqrt{2}m_3$	$m_2 = m_4 = 0$
Case 3	$m_1 \pm m_2$	$m_3 = m_4 = 0$

3.2 Dirac spinor?

- In 2-dim. continuum theory, $\pi/2$ rotational transf. of a Dirac spinor

$$\psi'(x) = e^{i\frac{\pi}{4}\frac{[\gamma_1,\gamma_2]}{2i}}\psi(R(x)) = e^{i\frac{\pi}{4}\sigma_3}\psi(R(x))$$

- In 2-dim. our lattice theory, $\pi/2$ rotational transf. of a $SO(4)$ spinor

$$\Psi'(N) = V_{12}\Psi(R(N))$$

Case1 and 2 $\cdots$ the spinor property of a continuum theory

	Diagonalized V_{12}	Phase factor of V_{12}
Case 1 ○	$\begin{pmatrix} e^{i\frac{\pi}{4}\sigma_3} & 0 \\ 0 & e^{i\pi}(e^{i\frac{\pi}{4}\sigma_3})^\dagger \end{pmatrix}$	$e^{i\vartheta} = e^{i\pi/2} = i$
Case 2 ○	$\begin{pmatrix} e^{i\frac{\pi}{4}\sigma_3} & 0 \\ 0 & e^{i\pi}(e^{i\frac{\pi}{4}\sigma_3})^\dagger \end{pmatrix}$	$e^{i\vartheta} = e^{i\pi} = -1$
Case 3 ×	$\begin{pmatrix} (e^{i\frac{\pi}{4}\sigma_3})^2 & 0 \\ 0 & e^{i\pi/2}\{(e^{i\frac{\pi}{4}\sigma_3})^\dagger\}^2 \end{pmatrix}$	$e^{i\vartheta} = e^{-i\pi/4} = -\frac{1+i}{\sqrt{2}}$

3.3 Pole analysis

- 2-dim. Staggered Dirac operator $(2a = 1, U = 1, -\pi < p_\mu < \pi)$

$$\begin{aligned} D_{st} &= \sum_{\mu,\vec{\varepsilon}} (\Gamma_{\mu,\vec{\varepsilon}})_{(r,r')} (D_{\mu,\vec{\varepsilon}})_{(N,N')} \\ &= \sum_\mu \frac{1}{2} \left\{ \gamma_\mu (\nabla_\mu^- + \nabla_\mu^+) + i\tilde{\gamma}_\mu (\nabla_\mu^- - \nabla_\mu^+) \right\} \\ &\overset{\text{Fourier transf.}}{\Longrightarrow} \sum_\mu \left\{ i\gamma_\mu \sin p_\mu + i\tilde{\gamma}_\mu (1 - \cos p_\mu) \right\} \\ &\equiv D_{st}(p) \end{aligned}$$

where $\nabla_\mu^- = \delta_{N,N'} - \delta_{N-\hat{\mu},N'}$, $\nabla_\mu^+ = \delta_{N+\hat{\mu},N'} - \delta_{N,N'}$.

- $\mathcal{O}_2, \mathcal{O}_3, \mathcal{O}_4$ do not commute with a staggered Dirac operator D_{st}

$\Longrightarrow$ **Pole analysis**

From $\Gamma_3 = i\gamma_1\gamma_2$, $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, $\{\tilde{\gamma}_\mu, \tilde{\gamma}_\nu\} = 2\delta_{\mu\nu}$, $\{\gamma_\mu, \tilde{\gamma}_\nu\} = 0$,

$$\begin{aligned} [D_{st}, \mathcal{O}_2] &= [D_{st}, \Gamma_3] \neq 0, \\ [D_{st}, \mathcal{O}_3] &= [D_{st}, \tilde{\gamma}_1 + \tilde{\gamma}_2] \neq 0, \\ [D_{st}, \mathcal{O}_4] &= [D_{st}, \Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)] \neq 0. \end{aligned}$$

- **2-dim. Staggered Dirac operator (momentum space)**

$$D_{st}(p) = \begin{pmatrix} 0 & is_2+c_2 & is_1+c_1 & 0 \\ is_2-c_2 & 0 & 0 & is_1+c_1 \\ is_1-c_1 & 0 & 0 & -is_2-c_2 \\ 0 & is_1-c_1 & -is_2+c_2 & 0 \end{pmatrix}$$

$s_i = \sin p_i \quad , \quad c_i = 1-\cos p_i \quad (i=1,2)$

- **Improved Staggered Dirac operator (momentum space): $D^{Imp}(p)$**

$$\begin{cases} \text{Case 1} \cdots D_1^{Imp}(p) \equiv D_{st}(p) + m_1 1 + m_4 \Gamma_3(\tilde{\gamma}_1+\tilde{\gamma}_2) \\ \text{Case 2} \cdots D_2^{Imp}(p) \equiv D_{st}(p) + m_1 1 + im'_3(\tilde{\gamma}_1+\tilde{\gamma}_2) \\ \text{Case 3} \cdots D_3^{Imp}(p) \equiv D_{st}(p) + m_1 1 + im'_2 \Gamma_3 \end{cases}$$

mass parameters $m_1, m'_2 = -im_2, m'_3 = -im_3, m_4$: real

	Pole masses	Parameter condition
Case 1 ○	2-fold degeneracy ×2	$-1 < m_4 < 1$
Case 2 ×	4-fold degeneracy	—
Case 3 ○	2-fold degeneracy ×2	$-1 < m'_2 < 1$

4. Summary and future plans

Summary

- $SO(2D)$ **Clifford algebra formulation**
- 4-candidates of improved operator

$$1, \quad \Gamma_3, \quad \tilde{\gamma}_1 + \tilde{\gamma}_2, \quad \Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)$$

Good candidate of the light single Dirac mode $\cdots$ Case 1

	Improved operator	Mass matrix	Dirac spinor	Pole
Case 1	$1, \ \Gamma_3(\tilde{\gamma}_1 + \tilde{\gamma}_2)$	○	○	○
Case 2	$1, \ \tilde{\gamma}_1 + \tilde{\gamma}_2$	○	○	×
Case 3	$1, \ \Gamma_3$	○	×	○

Future plans

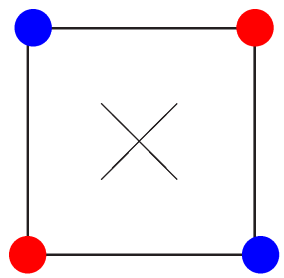
- Analysis in interaction theory
- Analysis in 4-dimensional lattice theory

Single taste reduction

	Improved operator	Mass matrix	Dirac spinor	Pole
Case 1	$1,\ \Gamma_3(\tilde{\gamma}_1+\tilde{\gamma}_2)$	○	○	○
Case 2	$1,\ \tilde{\gamma}_1+\tilde{\gamma}_2$	○	○	×
Case 3	$1,\ \Gamma_3$	○	×	○

- Case 1 $\cdots$ $m_1^2 = 2m_4^2$ and $m_4 \to \pm 1$

Heavy mode $\Longrightarrow$ ∞ mode



For infinite mode,
even site fermion decouples odd site fermion.