Representation of spacetime diffeomorphisms in canonical geometrodynamics under harmonic coordinate conditions

C L Stone and K V Kuchař
Department of Physics, University of Utah, Salt Lake City, UT 84112, USA

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Abstract. Using a method developed by Isham and Kuchař, the Lie algebra of the spacetime diffeomorphism group is mapped homomorphically into the Poisson algebra of dynamical variables on the phase space of canonical geometrodynamics. To accomplish this the phase space must first be extended by including embedding variables and their conjugate momenta. Also, the relationship between the embeddings and the spacetime metric must be limited by coordinate conditions. This is done by adding a source term to the Hilbert action of general relativity in such a way that the standard vacuum Einstein equations result when harmonic conditions are imposed as a supplementary set of constraints. In the process the usual super-Hamiltonian and supermomentum constraints of geometrodynamics become temporarily altered, but they are restored to their familiar form at the end by imposing the harmonic constraints, constraints that are preserved in the dynamical evolution generated by the total Hamiltonian. It is shown that the same method can also be applied to other generally covariant systems, namely the free relativistic particle and the bosonic string.

1. Introduction

A supposed disadvantage of the Hamiltonian formulation of a field theory is its alleged lack of spacetime covariance, since one particular observer’s time coordinate seems to be accorded privileged status. This can be especially troubling for those who aspire to quantize canonical geometrodynamics. On the one hand the quantization of a field theory proceeds most naturally from its canonical version, the special role of time not presenting any difficulty in quantum mechanics. On the other hand Einsteinian gravity is of all field theories the most self-consciously ‘relativistic’, and any singling out of one particular time variable seems both unnatural and unwarranted. Although at a deeper level this may be telling us something fundamental about any possible quantum theory of gravity, the discrepancy between the ‘covariant’ and ‘canonical’ aspects of geometrodynamics is to a large extent only apparent. The issue has been analysed by Isham and Kuchař (1985b). They have shown how the Lie bracket algebra $\mathcal{L}_{\text{Diff}\, \mathcal{M}}$ of the generators of spacetime diffeomorphisms (those generators being complete spacetime vector fields) can be mapped homomorphically into the Poisson bracket algebra of certain functions of canonical data on the (extended) geometrodynamical phase space. That is, although canonical geometrodynamics does not exhibit manifest spacetime covariance, the spacetime diffeomorphism group is merely hidden, not really lost, in the $3+1$ Hamiltonian treatment of general relativity. The elucidation of this issue is of particular importance for a group-theoretical approach to quantum gravity (Isham 1984, section 6).
The key step in the Isham–Kuchař method of representing spacetime diffeomorphisms is to enlarge the phase space of the theory by including embedding variables and their conjugate momenta as part of the canonical data. The embeddings tell us specifically how the spatial 3-manifold (i.e. a spacelike hypersurface) is implanted in the spacetime 4-manifold. A foliation consisting of a one-parameter family of such non-intersecting spacelike hypersurfaces filling spacetime establishes an explicit connection between spacetime and the Cauchy surfaces of our Hamiltonian theory, and thereby turns out to be of crucial importance in the representation of spacetime diffeomorphisms in the canonical context.

The process of ensuring that a field theory on a fixed pseudo-Riemannian background be invariant under spacetime diffeomorphisms, by means of the aforementioned adjoining of the integration variables (sometimes called ‘surface variables’, particularly in the work of Dirac) to the configuration variables of the field, has been known for a long time. A field theory on such an extended configuration space, or its canonical version on the corresponding extended phase space, is called a parametrized field theory (Dirac 1951, 1964, Synge 1960, Lanczos 1970, Kuchař 1976b (section 7)). A characteristic of such generally covariant, or parametrization-invariant—or ‘homogeneous’ in mathematical terminology, as in Rund (1966)—theories is the presence of Hamiltonian constraints, instead of a true Hamiltonian, in their canonical versions (Dirac 1951, 1964, Hanson et al 1976, Sundermeyer 1982). No matter which particular field is being parametrized, the ‘projected’ forms of these constraints always exhibit certain common features which, interestingly enough, they share with the super-Hamiltonian and supermomentum constraints of canonical geometrodynamics. Because of this similarity, a variant of the Isham–Kuchař procedure for representing spacetime diffeomorphisms in parametrized Hamiltonian field theories (Isham and Kuchař 1985a) can also be used in canonical geometrodynamics (Isham and Kuchař 1985b).

The dynamic character of the metric in general relativity is in marked contrast to its static role in a parametrized field theory on a fixed geometrical background. As a result, in order for the Isham–Kuchař method to be adapted successfully to canonical geometrodynamics it is necessary not only to extend the phase space of the theory by the explicit inclusion of the embeddings and their conjugate momenta (i.e. to parametrize), but also to lock the embedding variables and the metric variables together by means of (covariantly written) ‘coordinate conditions’. Isham and Kuchař chose to use Gaussian coordinate conditions, since they are algebraic in the spacetime metric. A question raised in their work is whether other coordinate conditions can be used instead to represent $LDiffM$ in canonical geometrodynamics. (A brief discussion of coordinate conditions in relativity, together with further references, is given in footnote 18 of Kuchař and Torre (1991a).)

In this paper we are concerned, in particular, to address how the Isham–Kuchař procedure can be implemented under the (non-algebraic) de Donder (or ‘harmonic’) coordinate conditions (de Donder 1921, Lanczos 1923). Not only are harmonic coordinates useful in the study of gravitational radiation, as well as often being favoured in the path-integral approach to quantum gravity, but also some physicists, most notably Fock (1964), have regarded them as being in some sense ‘preferred’ coordinates for Einsteinian spacetimes, similar to the way in which Minkowski coordinates are preferred coordinates in the flat spacetime of special relativity. Furthermore, they are of great importance in the Cauchy problem of general relativity, largely due to the work of Choquet-Bruhat, who showed that it is sufficient to solve the gravitational
initial value problem in harmonic coordinates (e.g. see Bruhat 1962). The de Donder conditions in general relativity are a descendant of the Lorentz gauge condition in electrodynamics. In an earlier paper (Kuchař and Stone 1987) we demonstrated how to use the Isham–Kuchař method to represent spacetime diffeomorphisms in the canonical version of the parametrized Maxwell field subject to the Lorentz gauge. Here we intend to show how harmonic coordinate conditions can indeed be used to represent $L\text{Diff}M$ in canonical geometrodynamics. In their original paper Isham and Kuchař employed a special fixed foliation with respect to which the spacetime metric had the Gaussian form. Here, however, we shall instead introduce the harmonic coordinate conditions 'dynamically', by adding a parametrized Fermi-type term to the usual Hilbert action of general relativity in such a way that if the harmonic 'gauge' is imposed as a constraint the standard vacuum Einstein equations will result. (Taking a different tack, Kuchař and Torre (1991b) handle the harmonic conditions in canonical geometrodynamics through Lagrange multipliers. Unlike the Gaussian conditions, the harmonic conditions lead to dynamical multipliers and hence to a double extension of the phase space.)

This paper is organized as follows. In section 2 we treat the gravitational action modified for harmonic coordinates, finding that in the process of parametrizing this action we encounter a second-order Lagrangian. In section 3 we reduce the second-order Lagrangian to first-order form by introducing a new set of field variables and then put the action of parametrized harmonic gravity into canonical form. In section 4 we use the Isham–Kuchař method to represent $L\text{Diff}M$ in this theory and show how standard geometrodynamics can be recovered in our canonical scheme. The formalism developed in this paper (adding an appropriate harmonic term and thereby breaking the diffeomorphism invariance of the theory, parametrizing to restore that invariance so as to be able to use the Isham–Kuchař procedure, handling the Hamiltonization of the resulting second-order Lagrangian) is sufficiently general to be applicable to the representation of diffeomorphisms under harmonic conditions in other generally covariant systems. As illustrations of this we include two appendices dealing with systems having mathematical structures that are somewhat similar to, albeit simpler than, that of general relativity. Specifically, we consider how to represent diffeomorphisms under harmonic coordinate conditions in the extended phase spaces of the free relativistic particle in appendix A and the closed bosonic string in appendix B.

2. The gravitational action and harmonic coordinates

In the canonical treatment of the evolution of a field on a prescribed pseudo-Riemannian spacetime background, the spatial 3-manifold $\Sigma$, equipped with coordinates $x^a$, is embedded in the spacetime 4-manifold $M$, equipped with coordinates $X^A$, by the mapping

$$X^A = X^A(x^a). \quad (1)$$

The spacetime (upper-case latin) indices take values 0, 1, 2, 3, whereas the spatial (lower-case latin) indices take values 1, 2, 3. We assume that the hypersurface (1) is spacelike with respect to the spacetime metric $g_{AB}(X^C)$ on $M$, where $g_{AB}$ has
signature \((-, +, +, +)\). At each point of the hypersurface we can construct the
(anholonomic) spacetime basis consisting of three vectors
\[
X^A_a(x; X) := X^A_a(x) := \frac{\partial X^A(x)}{\partial x^a}
\] (2)
tangent to the hypersurface, and the future-pointing unit normal \(n^A(x; X)\) defined
by the relations
\[
g_{AB} X^A_a n^B = 0 \quad \text{and} \quad g_{AB} n^A n^B = -1.
\] (3)
The notation \((x; X)\) means that \(X^A_a\) and \(n^A\) are not only functions of the spatial
point \(x\) but are also functions of the embedding \(X\). Spacetime indices are lowered
or raised, respectively, by the spacetime metric \(g_{AB}\) and its inverse \(g^{AB}\). Spatial
indices are lowered by the (positive definite) spatial metric
\[
\gamma_{ab}(x; X) := g_{AB}(X(x)) X^A_a(x; X) X^B_b(x; X)
\] (4)
induced by projecting \(g_{AB}\) onto the hypersurface, and they are raised by its inverse
\(\gamma^{ab}(x; X)\).
To describe a continuous deformation of the hypersurface through spacetime,
we incorporate it into a foliation \(X^A(t, x^a)\) of such spacelike hypersurfaces labelled by
a parameter \(t\). The spacetime deformation vector
\[
N^A := \dot{X}^A := \frac{\partial X^A(t, x^a)}{\partial t}
\] (5)
connects those events on neighbouring hypersurfaces that have the same spatial
coordinates \(x^a\). The lapse-shift decomposition consists of expressing the deformation
vector \(N^A\) in terms of its components \(N\) (the 'lapse' function) and \(N^a\) (the 'shift'
vector) with respect to the basis \(n^A, X^A_a:\)
\[
N^A = N n^A + N^a X^A_a
\] (6)
where
\[
N := -n_A N^A =: N^\perp \quad \text{and} \quad N^a := X^a_A N^A.
\] (7)Here
\[
n_A := g_{AB} n^B \quad \text{and} \quad X^a_A := g_{AB} X^B_b \gamma^{ab}
\] (8)
are the spacetime hypersurface cobasis covectors dual to \(n^A\) and \(X^A_a\). Notice from
(7) that the lapse and shift are both linear in the embedding velocities \(X^A = N^A\).
Now let \(X^A(x^a)\) be four independent harmonic scalar functions of the generic
spacetime coordinates \(x^a\), and define
\[
\Gamma^A(x^\gamma; g_{\alpha\beta}, X^A) := \Box X^A(x^\gamma) := g^{-1/2} g^{1/2} g^{\alpha\beta} X^A_\beta, \alpha = 0
\] (9)
with
\[
g := \det(g_{\alpha\beta}) \quad \text{and} \quad X^A_\beta := \frac{\partial X^A(x)}{\partial x^\beta}.
\]
When we use the scalars $X^A$ as coordinates in our spacetime manifold $M$, that is, when we put $x^\alpha = \delta^\alpha_A X^A$, (9) turns into a set of four conditions on the metric $g_{AB}(X^C)$:

$$\Gamma^A (X^B; g_{CD}) := \left| \det(g_{CD}) \right|^{-1/2} \left( | \det(g_{CD}) |^{1/2} g^{AB},_B \right) = 0. \tag{10}$$

These are the well-known harmonic coordinate conditions.

We modify the Hilbert action

$$S^G[g_{AB}] := \int_M d^4 X \left| \det(g_{AB}) \right|^{1/2} R[g_{AB}] \tag{11}$$

of Einsteinian gravity by adding to it the term

$$S^H[g_{AB}] := \frac{1}{2} \int_M d^4 X \left| \det(g_{CD}(X)) \right|^{1/2} g_{AB}(X) \Gamma^A (X; g_{CD}) \Gamma^B (X; g_{CD}) . \tag{12}$$

Because $\Gamma^A (X^B; g_{CD})$ transforms as a vector only under affine transformations of $X^A$, (12) breaks the diffeomorphism invariance of the Hilbert action (11). The field equations following from the variation of the total action

$$S[g_{AB}] := S^G[g_{AB}] + S^H[g_{AB}] \tag{13}$$

are no longer the standard vacuum Einstein equations. However, because the 'harmonic action' (12) is quadratic in $\Gamma^A (X^B; g_{CD})$, these field equations reduce to the Einstein equations after the harmonic conditions (10) are imposed. This trick goes back at least to Fermi (1929, 1930, 1932), who used it for treating the electromagnetic field under the Lorentz gauge condition. We call the system (13) harmonic gravity.

The diffeomorphism invariance of the action can be restored by its parametrization (Dirac 1951, 1964, Synge 1960, Lanczos 1970, Kuchar 1976b (section 7)). We express the privileged harmonic coordinates $X^A$ as functions of arbitrary label coordinates $x^\alpha$,

$$X^A = X^A(x^\alpha) \tag{14}$$

and adjoin these functions to the metric variables. We then define the parametrized action $S[g_{\alpha\beta}, X^A]$ by the requirements that it be invariant under transformations of $x^\alpha$ and reduce to the old action (13) when the harmonic coordinates are used as the labels:

$$S[g_{\alpha\beta}, X^A = \delta^\alpha_A x^\alpha] = S[g_{AB}] . \tag{15}$$

These two requirements uniquely determine $S[g_{\alpha\beta}, X^A]$. Of course the Hilbert action (11) is invariant as it stands, and its parametrized form does not depend upon $X^A(x^\alpha)$:

$$S^G[g_{\alpha\beta}, X^A] = S^G[g_{\alpha\beta}] = \int_M d^4 x \left| g \right|^{1/2} R[g_{\alpha\beta}] . \tag{16}$$

A comparison of (9) with (10) reveals that the parametrized harmonic action $S^H$ is

$$S^H[g_{\alpha\beta}, X^A] = \frac{1}{2} \int_M d^4 x \left| g \right|^{1/2} g_{\alpha\beta} X^A \delta^\alpha X^B \Gamma^A (x; g_{\gamma\delta}, X^D) \Gamma^B (x; g_{\gamma\delta}, X^D) . \tag{17}$$

The diffeomorphism invariance of the parametrized action $S[g_{\alpha\beta}, X^A]$ ensures that the equations of motion obtained by varying $X^A(x^\alpha)$ follow from those obtained by varying the metric $g_{\alpha\beta}(x^\gamma)$.
3. Canonical description of harmonic gravity

To engage in canonical geometrodynamics we need to pass to the Hamiltonian form of the parametrized action

$$S[g_{\alpha\beta}, X^A] = S^G[g_{\alpha\beta}] + S^H[g_{\alpha\beta}, X^A].$$

(18)

The diffeomorphism invariance of the action implies that the evolution of the dynamical variables is governed by the familiar super-Hamiltonian and supermomentum constraints. The Hamiltonization of the Hilbert action presents no obstacles, and it can be handled by the standard Dirac–ADM algorithm. The canonical form of the Dirac–ADM action is

$$S^G[\gamma_{ab}, p^{ab}; N, N^a] = \int dt \int d^3x \left( p^{ab} \dot{\gamma}_{ab} - N H^G - N^a H^G_a \right)$$

(19)

where the gravitational super-Hamiltonian $H^G$ and the gravitational supermomentum $H^G_a$ are

$$H^G = \gamma^{-1/2}(p_{ab} p^{ab} - \frac{1}{2} p^2) - \gamma^{1/2} R[\gamma_{ab}]$$

(20)

and

$$H^G_a = -2 p^b_{ab}$$

(21)

with $p := p^{ab} \gamma_{ab}$ and $\gamma := \det(\gamma_{ab})$. An inspection of the harmonic term (17), however, along with (9), reveals an unexpected difficulty: the action (17) depends upon the second derivatives of the foliation $X^A(x^\alpha) = X^A(t, x^a)$ with respect to the label time. We propose to handle these second derivatives by introducing new variables $Y^A$ such that $X^A$ can be replaced by $\dot{Y}^A$. Then it will be possible to rewrite our action in first-order form in terms of $\dot{X}^A$ and $Y^A$. We shall enforce the identification of $Y^A$ with $\dot{X}^A$ by means of Lagrange multipliers and identify the momenta $P_A$ and $\Pi_A$ canonically conjugate to $X^A$ and $Y^A$ respectively.

The easiest way of achieving our aim is to introduce a new field variable

$$Y^{A\alpha} := |g|^{1/2} g^{\alpha\beta} X^A_{\beta}.$$  

(22)

Note that $Y^{A\alpha}$ is a spacetime vector density. The harmonic Lagrangian of (17) then takes the form

$$L^H = \frac{1}{2} |g|^{-1/2} g_{AB} Y^{A\alpha} Y^{B\beta}.$$  

(23)

In (23), $g_{AB}$ is to be expressed as a function of $g_{\alpha\beta}$ and $Y^{A\alpha}$. We enforce the relationship (22) between $X^A$ and $Y^{A\alpha}$ with a Lagrange multiplier $\Lambda_A^{\alpha}$:

$$L^A := \Lambda_A^{\alpha}(X^A_{\alpha} - |g|^{-1/2} g_{\alpha\beta} Y^{A\beta}).$$

(24)

(Like $Y^{A\alpha}$, $\Lambda_A^{\alpha}$ is a spacetime vector density.)
The total Lagrangian of parametrized harmonic gravity is the sum

\[ L = L^G + L^{HA} \quad \text{where} \quad L^{HA} := L^H + L^A \]  

(25)

and \( L^G \) is the usual gravitational Lagrangian. When written in terms of the new variables \( \Lambda_A^{\alpha} \), \( X^A \), \( Y^{A\alpha} \), and \( g_{\alpha\beta} \), \( L^{HA} \) does not depend upon the derivatives of the metric. The canonical form of harmonic gravity therefore follows the pattern of all theories with non-derivative gravitational coupling (Kuchař 1976b (section 11)), being considerably different from that of theories derivatively coupled to gravity (Kuchař 1977). The Hamiltonian \( H(N, \bar{N}) \) is a linear combination of the super-Hamiltonian and supermomentum constraints:

\[ H(N, \bar{N}) = \int_{\Sigma} d^3 x \left( N H + N^a H_a \right). \]  

(26)

The constraints following from the variation of the lapse and the shift in the total action are additive. The super-Hamiltonian \( H \) is obtained by adding to the gravitational super-Hamiltonian \( H^G \) of (20) the energy density \( H^H \) of the harmonic fields:

\[ H := H^G + H^H \approx 0. \]  

(27)

Similarly, the supermomentum \( H_a \) is the sum

\[ H_a := H^G_a + H^H_a \approx 0 \]  

(28)

of the gravitational supermomentum \( H^G_a \) given by (21) and the harmonic momentum density \( H^H_a \). In (27) and (28) the symbol \( \approx \) means 'is weakly equal to'. That is, the quantity formed from the canonical data does not vanish identically, but only modulo the constraints. The symbol \( \approx \) also serves to remind us that we must work out the values of any Poisson brackets involving constrained quantities before imposing the constraints. Once we identify the conjugate pairs describing the harmonic field, the form of the momentum density \( H^H_a \) is fixed by the requirement that it generate the changes of the canonical variables under \( \text{Diff}\Sigma \). Therefore we do not need to worry about calculating \( H^H \), and, for the purpose of obtaining \( H^H \), we can put \( N^a = 0 \) in all steps of the calculation. This simplifies the ADM expressions for the spacetime metric to

\[ g_{\alpha\alpha} = -N^2 \quad g_{\alpha a} = 0 \quad g_{ab} = \gamma_{ab} \quad \left( | g |^{1/2} = N \gamma^{1/2} \right). \]  

(29)

For \( N^a = 0 \), \( L^{HA} \) reduces to

\[ L^{HA} = P_a \dot{X}^A + N \gamma^{-1/2} Y^A P_a + \frac{1}{2} | g |^{-1/2} g_{AB} (\dot{Y}^A + Y^{A\alpha}) (\dot{Y}^B + Y^{B\beta}) \]

\[ + \Lambda_A^{\alpha} (X^A_a - N^{-1} \gamma^{-1/2} \gamma_{ab} Y^{A\beta}). \]  

(30)

The form of (24) helped us to identify \( P_a := \Lambda_a^{\alpha} \) with the momentum conjugate to \( X^A \). In addition to \( X^A \), the only quantity whose time derivative appears in (30) is \( Y^{A\alpha} := Y^{A\alpha} \). The momentum conjugate to \( Y^A \) is

\[ \Pi^A := \frac{\partial L^{HA}}{\partial \dot{Y}^A} = | g |^{-1/2} g_{AB} (\dot{Y}^B + Y^{B\beta}) \]  

(31)
This leads to the harmonic piece \( H^H(N) \) of the Hamiltonian (26):

\[
H^H(N) := \int \Sigma \left( P_A \dot{X}^A + \Pi_A \dot{Y}^A - L^{HA} \right)
= \int \Sigma d^3x \left[ -N\gamma^{-1/2}Y^AP_A + \frac{1}{2} \left| \gamma^{1/2}g^{AB}\Pi_A\Pi_B + Y^{Aa}\Pi_{A,a} \right. \right.
\left. - \Lambda_{A,a} \left( X^A_{,a} - N^{-1}\gamma^{-1/2}\gamma_{ab}Y^{Ab} \right) \right]. \tag{32}
\]

At this stage we vary \( \Lambda_{A,a} \) and obtain the equation

\[
Y^{Aa} = N\gamma^{1/2}\gamma_{ab}X^A_{,b} \tag{33}
\]

which enables us to eliminate both \( Y^{Aa} \) and \( \Lambda_{A,a} \) from the action. Comparison of (32) with (26) yields the harmonic energy density

\[
H^H = -\gamma^{-1/2}Y^AP_A + \gamma^{1/2}X^{Aa}\Pi_{A,a} + \frac{1}{2} \gamma^{1/2}g^{AB}\Pi_A\Pi_B. \tag{34}
\]

Here the metric \( g^{AB} \) is to be expressed in terms of the configuration variables \( X^A, Y^A \), and \( \gamma_{ab} \):

\[
g^{AB} = |g|^{-1}g^{\alpha\beta}Y^{A\alpha}Y^{B\beta} = -\gamma^{-1}Y^AY^B + \gamma^{ab}X^A_{a}X^B_{b}. \tag{35}
\]

The pairs of conjugate variables describing the harmonic fields are \( X^A, P_A \) and \( Y^A, \Pi_A \). With respect to \( \text{Diff}\Sigma \), \( X^A \) and \( \Pi_A \) behave as scalars, and \( Y^A \) and \( P_A \) as scalar densities. The momentum density \( H^H_a \) must generate the changes of these variables under \( L\text{Diff}\Sigma \). This requirement dictates the functional form of \( H^H_a \) (Dirac 1951, Kuchař 1976a (section 5)):

\[
H^H_a = X^A_{,a}P_A - Y^A\Pi_{A,a}. \tag{36}
\]

The energy density (34) and the momentum density (36) complete the gravitational expressions (20) and (21) into the super-Hamiltonian (27) and the supermomentum (28) of harmonic gravity.

4. Canonical representation of \( L\text{Diff}M \)

As in Gaussian gravity (Isham and Kuchař 1985b, Kuchař 1986, Kuchař and Torre 1991a), the constraints \( H \approx 0 \) and \( H_a \approx 0 \) given by (27) and (28) are linear in the momentum \( P_A \) conjugate to the embedding:

\[
H = n^AP_A + \bar{H} \quad \bar{H} := \gamma^{1/2}X^{Aa}\Pi_{A,a} + \frac{1}{2} \gamma^{1/2}g^{AB}\Pi_A\Pi_B + H^G
\]

\[
H_a = X^A_{,a}P_A + \bar{H}_a \quad \bar{H}_a := -Y^A\Pi_{A,a} + H^G_a. \tag{37}
\]

Here the coefficient

\[
n^A = -\gamma^{-1/2}Y^A \tag{38}
\]
is the unit normal to the embedding \(X^A(x)\) in the harmonic system of coordinates \(X^A\).

The linearity of the constraints (37) is instrumental in applying the Isham–Kuchař algorithm to harmonic gravity. It enables us to replace the projected constraints \(H\) and \(H_a\) of (37) by the unprojected ones \(H_A\), where

\[
H_A(x) := P_A(x) - \hat{H}(x; X^B, Y^B, \Pi_B, \gamma_{ab}, p^{ab}) \ n_A(x; X^B, Y^B) + \hat{H}_a(x; Y^B, \Pi_B, \gamma_{ab}, p^{ab}) \ X_A^a(x; X^B, Y^B).
\]  

(39)

Here \((-n_A, X_A^a)\) is the cobasis to the normal basis \((n^A, X_A^a)\):

\[
n_A = -\frac{1}{3!} \delta_{ABCD} X^B_b X^C_c X^D_d \delta^{bcd}/\Delta
\]

\[
X_A^a = \frac{1}{2!} \delta_{ABCD} X^B_b X^C_c n^D \delta^{abc}/\Delta
\]

\[
\Delta := \frac{1}{3!} \delta_{ABCD} n^A X^B_b X^C_c X^D_d \delta^{bcd}.
\]  

(40)

In the unprojected constraints (39), \(P_A(x)\) is cleanly separated from the rest of the canonical variables. As a consequence, the Poisson brackets between them vanish strongly:

\[
\{H_A(x), H_B(x')\} = 0.
\]  

(41)

One can show this by a straightforward argument that circumvents much tedious algebra (Kuchař and Torre 1991a, section 6): The old constraints (37) close according to the Dirac ‘algebra’ (Dirac 1951), and as a consequence their Poisson brackets vanish weakly. The new constraints (39) are equivalent to the old constraints, and their Poisson brackets must therefore also vanish weakly. However, the form (39) of the new constraints ensures that the Poisson brackets (41) do not depend on the embedding momentum \(P_A(x)\). The constraints (39) therefore cannot help the brackets to vanish, which means that the brackets must vanish strongly.

Equation (41) and the separation of \(P_A(x)\) from the rest of the canonical variables in (39) mean that we can construct the Isham–Kuchař representation of spacetime diffeomorphisms as usual. Let \(U, V \in LDiffM\) be two generators of spacetime diffeomorphisms, that is, two complete spacetime vector fields on \(M\). Restrict them to the embeddings so that \(U(x^a; X^A) := U(X^A(x^a))\) and \(V(x^a; X^A) := V(X^A(x^a))\). Map these vector fields into total Hamiltonians \(H(U)\) and \(H(V)\) by the prescription

\[
U \mapsto H(U) := \int_\Sigma d^3x \ U^A(X^B(x)) \ H_A(x; \gamma_{ab}, p^{ab}, X^A, P_A, Y^A, \Pi_A)
\]  

(42)

with \(H_A(x)\) given by (39), and similarly for \(V\) and \(H(V)\). This mapping is a homomorphism from the Lie bracket algebra \(LDiffM\) of \(U\) and \(V\) into the Poisson bracket algebra of the dynamical variables \(H(U)\) and \(H(V)\) on the extended phase space of our gravitational theory, i.e.

\[
\{H(U), H(V)\} = H([UV]).
\]  

(43)
Here the Lie bracket \([UV]\) is defined as

\[
[UV] := -(U^B V_A^A - V^B U_A^A) \frac{\partial}{\partial X^A}
\]

(44)

which is just the negative of the familiar commutator \([U, V]\) of the two vector fields. The results (42) and (43) constitute a representation of \(L\text{Diff} M\) in canonical geometrodynamics under harmonic coordinate conditions.

Now all that remains is to ensure that our theory is equivalent to standard vacuum geometrodynamics. We can accomplish this in the canonical context simply by introducing some supplementary constraints. The primary one of these is the requirement that the special coordinates \(X^A\) satisfy the harmonic conditions (9) and (10). To make sure that they do, we impose the constraint

\[
C_1 := \Pi_A \approx 0.
\]

(45)

The requirement that this constraint hold for all time slices of our foliation, that is

\[
\dot{C}_1 = \{C_1, H(N, \overline{N})\} \approx 0
\]

leads, after invoking (45), to the secondary constraint

\[
C_2 := P_A \approx 0.
\]

(46)

Using (45) and (46) in (34) and (36), we see that \(H\) and \(H_a\) of (27) and (28) indeed reduce to the standard geometrodynamical super-Hamiltonian and supermomentum of (20) and (21). Furthermore, it is easy to confirm that once \(C_1\) and \(C_2\) are imposed on the initial data they are preserved in the dynamical evolution generated by our total Hamiltonian \(H(N, \overline{N})\), because

\[
\dot{C}_1 = \{C_1, H(N, \overline{N})\} \approx 0 \approx \{C_2, H(N, \overline{N})\} = \dot{C}_2.
\]

In other words, if the generators \(H(U)\) representing spacetime diffeomorphisms in (42) and (43) start evolving a point of the extended phase space which lies at the intersection of the constraint surfaces

\[
H(x) \approx 0 \approx H_a(x) \quad \text{and} \quad C_1(x) \approx 0 \approx C_2(x)
\]

(47)

the point will continue moving along this intersection for all later times.

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Appendix A. The free relativistic particle and temporal diffeomorphisms

To illustrate further the functioning of the formalism developed in this paper, we apply it to a system having only a finite number of degrees of freedom—specifically, to a single particle moving in Minkowski spacetime. The action of this theory, rather than being an integral over spacetime as in general relativity, will simply be an integral over time. Hence, instead of trying to find a phase space representation of the Lie algebra $\mathcal{L}\text{Diff} M$ of the spacetime diffeomorphism group, we are only concerned to represent the Lie algebra $\mathcal{L}\text{Diff}_R$ of diffeomorphisms of the real line. In lieu of three-dimensional space-like hypersurfaces embedded in a four-dimensional spacetime manifold $M$, we shall have points embedded in a one-dimensional 'time manifold' $\mathbb{R}$.

### A.1. The action

The particle moves in Minkowski spacetime equipped with the pseudo-Cartesian coordinates $z^i, i = 0, 1, 2, 3$, and the flat metric $\eta_{i\kappa} = \text{diag}(-1, 1, 1, 1)$. The time manifold $\mathbb{R}$ is parametrized by a single coordinate $x^0 = t$ and is equipped with a one-dimensional metric $g_{00}(x^0)$. The action functional whose extremization yields the particle’s worldline can be written in the form

$$S^P[g_{00}, z^i] = -\frac{1}{2} m \int_\mathbb{R} dt \ |g|^{1/2} \left(g_{00}z^0z^0_0\eta_{i\kappa} + 1\right). \quad (A1)$$

Here both the worldline $z^i(t)$ and the time metric $g_{00}(t)$ are to be varied. The particle action (A1) is invariant under $\text{Diff}\mathbb{R}$, and it is closely analogous to the Hilbert action of general relativity when we relate the variables as follows:

$$t = x^0 \quad \longleftrightarrow \quad x^\alpha \quad (g_{00}, z^i) \quad \longleftrightarrow \quad g_{\alpha\beta} \quad \longrightarrow \quad (N, N^\alpha) \quad z^i \quad \longleftrightarrow \quad \gamma_{\alpha\beta}. \quad (A2)$$

In particle theory the time metric $g_{00}$ is a Lagrange multiplier that is clearly separated from the dynamical variables $z^i$ of the particle.

In a one-dimensional space one can write

$$g_{00} = -N_0 N_0 \quad g_{00} = -N^0 N^0 \quad \text{and} \quad |g|^{1/2} = N_0 \quad (A3)$$

where $N^\alpha = (N_0)^{-1} = N_0^{-1}$. As the position of the index indicates, $N^\alpha$ is a vector and $N_0$ a covector under $\text{Diff}\mathbb{R}$. Scalar densities, like $|g|^{1/2}$, can be identified with covectors, like $N_0$. In terms of the variables (A3), the action (A1) takes the form

$$S^P[N_0, z^i] = \frac{1}{2} m \int_\mathbb{R} dt \left(N_0^{-1} \eta_{i\kappa} \dot{z}^i \dot{z}^\kappa - N_0\right) \quad (A4)$$

(with $\dot{z}^i := z^i_0$), as analysed by Hartle and Kuchař (1986, section II). The variation of (A4) with respect to $N_0$ yields

$$N_0 = \sqrt{-\eta_{i\kappa} \dot{z}^i \dot{z}^\kappa} = d(\text{proper time})/dt. \quad (A5)$$

By substituting (A5) back into (A4), we obtain the standard action

$$S[z^i] = -m \int_\mathbb{R} dt \sqrt{-\eta_{i\kappa} \dot{z}^i \dot{z}^\kappa} \quad (A6)$$
of a relativistic particle.

After introducing the momentum \( p_i \) canonically conjugate to \( z^i \), we cast the action (A4) into the Hamiltonian form

\[
S^P[N_0; \, z^i, \, p_i] = \int d\tau (p_i \dot{z}^i - N_0 H) \tag{A7}
\]

where

\[
H := \frac{1}{2m} (\eta^{\kappa\lambda} p_\kappa p_\lambda + m^2) \tag{A8}
\]

plays the role of the super-Hamiltonian, and \( H(N_0) := N_0 H \) that of the Hamiltonian of the particle. The variation of (A7) with respect to \( N_0 \) yields the Hamiltonian constraint \( \dot{H} \approx 0 \).

\[A.2. \text{The harmonic time}\]

Let \( X^0(x^0) = T(t) \) be a harmonic scalar function of the time coordinate \( t = x^0 \) on the time manifold \( \mathbb{R} \):

\[
\Gamma^0 (t; \, \dot{\tau}, \, \dot{T}) := \nabla X^0(x^0) = |g|^{-1/2} (|g|^{1/2} g^{00} T_{,0}),_0
\]

\[
= (-N^0 T_{,0}),_0 N^{0} = -N_0^{-1}(N^{-1}_0),_0 = 0 . \tag{A9}
\]

When we choose \( T \) as a coordinate in \( \mathbb{R} \), i.e. when we put \( t = T \), equation (A9) assumes the form

\[
\Gamma^0 (T; \, g_{00}) := -N_0^{-1}(N^{-1}_0),_0 = 0 . \tag{A10}
\]

The harmonic time condition (A10) therefore means that \( N_0(T) \) does not depend upon \( T \). When we use \( T \) to parametrize the worldline \( z^i(T) \) of the particle, equation (A5) tells us that \( T \) is proportional to the proper time. The harmonic time \( T \) is therefore an affine parameter of the particle's geodesic motion.

As in general relativity, let us break the Diff\( \mathbb{R} \) invariance of the action (A1) by adding to it the harmonic term

\[
S^H[g_{00}] := \frac{1}{2} \int_{\mathbb{R}} dT \left| \det(g_{00}(T)) \right|^{1/2} g_{00}(T) \, \Gamma^0 (T; \, g_{00}) \, \Gamma^0 (T; \, g_{00})
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}} dT \, N_0 [(N^{-1}_0),_0]^2 . \tag{A11}
\]

The total action \( S = S^P + S^H \) yields back the geodesic motion after we impose the harmonic time condition (A10).

Diffeomorphism invariance can be restored by parametrization. As in harmonic gravity, the parametrized action of the particle takes the form

\[
S[g_{00}, \, z^i] := S^P[g_{00}, \, z^i] + S^H[g_{00}, \, T] \tag{A12}
\]

with

\[
S^H[g_{00}, \, T] = \frac{1}{2} \int_{\mathbb{R}} dt \, g^{1/2} g_{00} \Gamma^0 (t; \, g_{00}, \, T) \, \Gamma^0 (t; \, g_{00}, \, T)
\]

\[
= -\frac{1}{2} \int_{\mathbb{R}} dt \, N_0 (N^{-1}_0 T_{,0})^{-1} \left[(-N^0 T_{,0}),_0 N^{0}\right]^2 . \tag{A13}
\]

We have written the parametrized harmonic term (A13) in such a way that its invariance with respect to Diff\( \mathbb{R} \) is manifest.
A.3. The canonical action

To bring the harmonic term (A13) into first-order form, we introduce a new variable

\[ Y := |g|^{1/2} g^{\alpha \beta} T_{\alpha \beta} = -N^0 T_{\alpha \beta} = -N_0^{-1} \dot{\mathcal{T}}. \]  

(A14)

The field \( Y(t) \) is a vector density or—which is the same thing in one dimension—a scalar with respect to Diff R. In terms of \( Y \), the harmonic Lagrangian of (A13) is

\[ L^H = -\frac{1}{2} (N_0)^{-1} Y^{-2} \dot{Y}^2. \]  

(A15)

We enforce the connection (A14) between \( \dot{T} \) and \( Y \) by adding to the Lagrangian the term

\[ L^A := P(\dot{T} + N_0 Y). \]  

(A16)

We have immediately identified the Lagrange multiplier \( P \) with the momentum conjugate to \( T \). The momentum \( \Pi \) conjugate to \( Y \) is

\[ \Pi := \frac{\partial L^H}{\partial \dot{Y}} = -N_0^{-1} Y^{-2} \dot{Y}. \]  

(A17)

The harmonic Lagrangian \( L^H + L^A \) yields the Hamiltonian

\[ H^H(N_o) := P \dot{\mathcal{T}} + \Pi \dot{Y} - (L^H + L^A) = -N_0 (YP + \frac{1}{2} Y^{-2} \Pi^2). \]  

(A18)

The particle action (A1) has the same canonical form (A7)–(A8) as before. When complemented by the parametrized harmonic term, it becomes

\[ S[T, P, Y, \Pi, z^i, p_i; N_0] = \int_{\mathbb{R}} dt \left( P \dot{\mathcal{T}} + \Pi \dot{Y} + p_i \dot{z}^i - N_0 H \right) \]  

(A19)

with

\[ H := -YP - \frac{1}{2} Y^{-2} \Pi^2 + \frac{1}{2m} (\eta^{i \kappa} p_i p_\kappa + m^2). \]  

(A20)

The variation of (A19) with respect to \( N_o \) gives the super-Hamiltonian constraint \( H \approx 0 \).

A.4. Representing LDiff R and recovering the standard particle theory

The constraint \( H \approx 0 \) can be resolved with respect to \( P \), i.e. it can be written in the form

\[ H_o := P + \bar{H} \approx 0 \text{ with } \bar{H} := \frac{1}{2} Y^{-3} \Pi^2 - \frac{1}{2m} Y^{-1} (\eta^{i \kappa} p_i p_\kappa + m^2). \]  

(A21)

With \( P \) separated from the remaining canonical variables, equation (A21) enables us to proceed with the Isham–Kuchař construction: Let \( U(T) \) and \( V(T) \) be two complete vector fields on the real line. Restrict these fields to the embedding \( T(t) \),
so that \( U(t; T) := U(T(t)) \) and \( V(t; T) := V(T(t)) \), and map them into the total Hamiltonians \( H(U) \) and \( H(V) \), with

\[
U \mapsto H(U) := \int_{\mathbb{R}} dt \, U(T(t)) \, H_0(t; p, Y, \Pi, \nu)
\]

(A22)

and similarly for \( H(V) \). Then, as in (43),

\[
\{H(U), H(V)\} = H([UV]).
\]

(A23)

These last two results constitute a representation of \( LD\text{iff} \mathbb{R} \) in the extended phase space of the relativistic particle under harmonic coordinate conditions.

In order to make sure that our theory is equivalent to the standard parametrized particle theory, we must impose a constraint enforcing the harmonic coordinate condition. This is accomplished by setting

\[
C_1 := \Pi \approx 0.
\]

(A24)

To ensure that this \( C_1 \) constraint be preserved for all times \( t \) by our Hamiltonian formalism, we require that

\[
\dot{C}_1 = \{\Pi, N_\nu H\} \approx 0
\]

(A25)

which leads, after using (A24), to the secondary constraint

\[
C_2 := P \approx 0.
\]

(A26)

As we see from (A20), imposition of the constraints (A24) and (A26) causes \( H \) to reduce to the standard parametrized particle super-Hamiltonian constraint given by (A8). Furthermore, because \( \dot{C}_1 = \{C_1, N_\nu H\} \approx 0 \approx \{C_2, N_\nu H\} = \dot{C}_2 \), both \( C_1 \) and \( C_2 \) are preserved in the dynamical evolution generated by the total Hamiltonian \( N_\nu H \) of (A19) and (A20).

Appendix B. The closed bosonic string and world sheet diffeomorphisms

As a third example of the use of our formalism we apply it to a system intermediate in complexity between the particle and general relativity. The string model involves an infinite number of degrees of freedom and—like gravity—it is an 'already parametrized' generally covariant theory possessing, in its canonical version, super-Hamiltonian and supermomentum constraints right from the beginning. At the same time, however, its mathematical form is significantly simpler than that of geometrodynamics.

A closed string is a one-dimensional object topologically equivalent to a circle. As it moves through a Minkowski spacetime of dimension \( D \) it sweeps out a two-dimensional cylindrical surface (topologically equivalent to \( \mathbb{R} \times S^1 \)) called the world sheet of the string. Owing to quantum mechanical considerations \( D \) is usually taken to be 26, but for our purposes we can leave it indeterminate. The one-dimensional string manifold \( S^1 \) embedded in the two-dimensional world sheet manifold \( m \) in this theory corresponds to the three-dimensional spacelike hypersurface \( \Sigma \) embedded in the four-dimensional spacetime manifold \( M \) in geometrodynamics. The representation of the Lie algebra \( LD\text{iff}m \) of the world sheet diffeomorphism group in the (extended) phase space of the closed bosonic string by means of the Isham–Kuchař procedure has previously been done (under harmonic conditions, among others) by Kuchař and Torre (1989). Their way of implementing the harmonic conditions differs somewhat from ours, however, and we shall here treat the problem by our method for the sake of illustration.
B.1. The standard action

The action functional for the free massless relativistic closed bosonic string is constructed so that the variational principle will minimize—or at least extremize—the area of the world sheet. The action can be taken to be (e.g. see Green et al (1987), section 1.3):

\[
S^S[g_{\alpha\beta}, z^\mu] := -\frac{1}{2} \int_{\mathcal{M}} \, d^2 x \, \sqrt{|g|} \, g^{\alpha\beta} n_{\mu\nu} z^\mu,_{\alpha} z^\nu,_{\beta}. \tag{B1}
\]

Here the lower case greek indices from the beginning of the alphabet (\(\alpha, \beta, \gamma\)) take on the values 0 or 1, and those from the middle part of the alphabet (\(\mu, \nu\)) run from 0 to \(D - 1\). The coordinates on the cylindrical world sheet manifold are \(x^\alpha := (t, x)\), where \(x\) (with \(0 \leq x \leq 2\pi\)) is a spatial coordinate and \(t\) is a temporal coordinate. Events on the world sheet have their positions in the \(D\)-dimensional Minkowski 'target space' specified (as functions of the parameters \(x\) and \(t\)) by the inertial coordinates \(z^\mu = z^\mu(x^\alpha) = z^\mu(t, x)\), and \(z^\mu,_{\alpha} := \partial z^\mu / \partial x^\alpha\). In (B1) \(g_{\alpha\beta}(t, x)\), with signature \((-+,+\)), is the metric of the world sheet manifold \(m\) over which we integrate in our action, \(g\) is its determinant, \(g^{\alpha\beta}\) is its inverse, and \(n_{\mu\nu} = \text{diag}(-1, 1, 1, \ldots)\) is the \(D\)-dimensional Minkowski metric. As the functional notation we have used serves to remind us, we must vary not only \(z^\mu\) but also the metric tensor \(g_{\alpha\beta}\) in the variational principle for the string. In this regard string theory is like geometrodynamics, in contrast to parametrized field theories on a fixed background. The string action (B1) is analogous to the Hilbert action of general relativity when we relate the variables as follows:

\[
x^\alpha = (t, x) \quad \longleftrightarrow \quad x^\alpha \quad \longleftrightarrow \quad g_{\alpha\beta}, \quad (g_{\alpha\beta}, z^\mu) \quad \longleftrightarrow \quad g_{\alpha\beta} \quad \longleftrightarrow \quad (N, N^a) \quad \longleftrightarrow \quad z^\mu \quad \longleftrightarrow \quad \gamma_{ab}. \tag{B2}
\]

To proceed in accordance with our earlier work, we introduce a foliation of embeddings of the one-dimensional string space \(S^1\) into the two-dimensional world sheet spacetime \(m\). We can then define a deformation vector \(N^a\), a tangent vector, an induced spatial metric \(\gamma_{ij}\) and its inverse \(\gamma^{ij}\), a normal vector \(n^a\), a lapse function \(N\), and a shift vector \(N^1\) in the familiar ways outlined at the beginning of section 2. Defining \(p_\mu\) to be the momentum conjugate to \(z^\mu\), we can cast the action (B1) into canonical form as

\[
S^S[z^\mu; p_\mu; N, N^1] = \int_{\mathbb{R}} dt \int_{S^1} \, dx \left( p_\mu \dot{z}^\mu - N H^S - N^1 H_1^S \right) \tag{B3}
\]

where

\[
H^S := \frac{1}{2} \gamma^{1/2} (\gamma^{-1} n_{\mu\nu} p_\mu p_\nu + 1) \tag{B4}
\]

and

\[
H_1^S := p_\mu z^\mu,_{1} \tag{B5}
\]

which are analogous to (20) and (21).
A feature that makes the string different from (and simpler than) gravity is that there are no derivatives of the metric $g_{\alpha\beta}$ occurring in the action (B1), in marked contrast to the Hilbert action (11), where $R[g_{AB}]$ contains both first and second derivatives of $g_{AB}$. As a result, the momentum $\pi^{11}$ conjugate to the induced string metric $\gamma_{11}$ vanishes identically and constitutes a primary constraint of the theory, i.e.

$$\pi^{11} = 0.$$  \hspace{1cm} (B6)

The freedom of the world sheet metric $g_{\alpha\beta}$ endows the lapse and shift in string theory with a concomitant freedom, similar to what it enjoy in geometrodynamics (and for the same reason). Varying the Langrange multipliers $N$ and $N^1$ in (B3) leads to the string super-Hamiltonian and supermomentum constraints

$$H^S(x) \approx 0 \approx H^S_1(x)$$  \hspace{1cm} (B7)

analogous to the super-Hamiltonian and supermomentum constraints in canonical geometrodynamics. The constraints (B7) arise prior to any parametrization process—i.e. we have not yet varied the foliation or defined a momentum conjugate to the embeddings—and this is a consequence (as in gravity) of the arbitrariness of the metric, in contrast to its fixed nature in parametrized field theories on a given Riemannian background. The Hamiltonian

$$H^S(N, N^1) := \int_{S^1} dx (N H^S + N^1 H^S_1)$$  \hspace{1cm} (B8)

is a linear combination of the constraints (B7) and therefore is also constrained to vanish weakly.

B.2. The action modified by adding a harmonic term and parametrizing

Let $X^A(x^\alpha)$ be two independent harmonic scalar functions of the generic world sheet coordinates $x^\alpha$ on $m$, and let us define

$$\Gamma^A(x^\gamma; g_{\alpha\beta}, X^A) := \Box X^A(x^\gamma) := g^{-1/2} g^{\alpha\beta} X^A_{,\alpha} = 0$$  \hspace{1cm} (B9)

where, as usual, $X^A_{,\beta} := X^A_{,\beta} := \partial X^A(x)/\partial x^\beta$. When we use the scalars $X^A$ as coordinates on $m$, that is, when we put $x^\alpha = \delta^\alpha_A X^A$, (B9) gives a pair of harmonic coordinate conditions on the metric $g_{AB}(X^C)$:

$$\Gamma^A(X^B; g_{CD}) := |\det(g_{DC})|^{-1/2} (|\det(g_{CD})|^{1/2} g^{AB})_{,B} = 0.$$  \hspace{1cm} (B10)

Proceeding as in section 2, we break the Diff$m$ invariance of the standard string action

$$S^S[g_{AB}, z^\mu] := -\frac{1}{2} \int_m d^2 X \ |\det(g_{CD})|^{1/2} g^{AB} n_{\mu\nu} z^\mu_{,A} z^\nu_{,B}$$  \hspace{1cm} (B11)

(cf (B1)) by adding to it the term

$$S^H[g_{AB}] := \frac{1}{2} \int_m d^2 X \ |\det(g_{CD})|^{1/2} g_{AB}(X) \Gamma^A(X; g_{CD}) \Gamma^B(X; g_{CD}) .$$  \hspace{1cm} (B12)
Diffeomorphisms and harmonic coordinates

The field equations following from the variation of the total action
\[ S[g_{AB}, z^\mu] := S^S[g_{AB}, z^\mu] + S^H[g_{AB}] \]  
are equivalent, after imposing the harmonic conditions (B10), to those following from the standard string action.

As before, diffeomorphism invariance of the action can be restored by parametrization. The parametrized string action has the form
\[ S[g_{\alpha\beta}, z^\mu, X^A] = S^S[g_{\alpha\beta}, z^\mu] + S^H[g_{\alpha\beta}, X^A] \]  
where the parametrized harmonic piece of the action is
\[ S^H[g_{\alpha\beta}, X^A] = \frac{1}{2} \int \, d^2 x \mid g \mid^{1/2} g_{\alpha\beta} X^\alpha_A X^\beta_B \Gamma^A(x; g_{\gamma\delta}, X^D) \Gamma^B(x; g_{\gamma\delta}, X^D). \]  

(B15)

B.3. The canonical form of the modified and parametrized action

To get the harmonic term (B15) into first-order form, we introduce the new field variable
\[ Y^{A\alpha} := \mid g \mid^{1/2} g^{\alpha\beta} X^A_{\beta}. \]  
Note that \( Y^{A\alpha} \) is a world sheet vector density. The harmonic Lagrangian of (B15) then takes the form
\[ L^H = \frac{1}{2} \mid g \mid^{-1/2} g_{AB} Y^{A\alpha}_{,\alpha} Y^{B\beta}_{,\beta}. \]  

(B17)

Here \( g_{AB} \) is to be expressed as a function of \( g_{\alpha\beta} \) and \( Y^{A\alpha} \). We enforce the relationship (B16) between \( X^A \) and \( Y^{A\alpha} \) with a Lagrange multiplier \( \Lambda^\alpha_A \):
\[ L^A := \Lambda^\alpha_A (X^A_{\alpha} - \mid g \mid^{-1/2} g_{\alpha\beta} Y^{A\beta}) \].  

(B18)

(\( Y^{A\alpha}, \Lambda^\alpha_A \) is a world sheet vector density.)

The total Lagrangian of the parametrized harmonic string is the sum
\[ L = L^S + L^{HA} \quad \text{where} \quad L^{HA} := L^H + L^A \]

(B19)

and \( L^S \) is the string Lagrangian of section B.1. The Hamiltonian \( H(N, N^1) \) is a linear combination of the super-Hamiltonian and supermomentum constraints:
\[ H(N, N^1) = \int_{S^1} dx (NH + N^1H_1). \]

(B20)

The constraints are additive; The super-Hamiltonian \( H \) is obtained by adding to the string super-Hamiltonian \( H^S \) of (B4) the energy density \( H^H \) of the harmonic fields:
\[ H := H^S + H^H \approx 0. \]

(B21)
Similarly, the supermomentum $H_1$ is the sum
\[ H_1 := H_{1}^{S} + H_{1}^{H} \approx 0 \] (B22)
of the string supermomentum $H_{1}^{S}$ given by (B5) and the harmonic momentum density $H_{1}^{H}$. As before, once we identify the conjugate pairs describing the harmonic field, the form of the momentum density $H_{1}^{H}$ is fixed by the requirement that it generate the changes of the canonical variables under $\text{Diff}S^1$. Therefore we do not need to worry about calculating $H_{1}^{H}$ and, for the purpose of obtaining $H^{H}$, we can put $N^1 = 0$ in all steps of the calculation. This simplifies the expressions for the world sheet metric (using the adapted foliation, as in the ADM treatment of gravity) to
\[ g_{00} = -N^2 \quad g_{01} = 0 \quad g_{11} = \gamma_{11} \quad (|g|^{1/2} = N \gamma^{1/2}). \] (B23)

For $N^1 = 0$, $L^{HA}$ reduces to
\[ L^{HA} = P_{A} \dot{X}^{A} + N \gamma^{-1/2} Y^{A} P_{A} + \frac{1}{2} |g|^{-1/2} g_{AB} (\dot{Y}^{A} + Y^{A,1}) (\dot{Y}^{B} + Y^{B,1}) \]
\[ + \Lambda_{A} (X^{A}_{1} - N^{-1} \gamma^{-1/2} \gamma_{11} Y^{A,1}). \] (B24)
As in section 3, $P_{A} := \Lambda_{A}^{o}$ is the momentum conjugate to $X^{A}$, and the momentum conjugate to $Y^{A} := Y^{A,0}$ is
\[ \Pi_{A} := \frac{\partial L^{HA}}{\partial \dot{Y}^{A}} = |g|^{-1/2} g_{AB} (\dot{Y}^{B} + Y^{B,1}, 1). \] (B25)

This leads to the harmonic piece $H^{H}(N)$ of the Hamiltonian (B20):
\[ H^{H}(N) := \int_{S^1} dx (P_{A} \dot{X}^{A} + \Pi_{A} \dot{Y}^{A} - L^{HA}) \]
\[ = \int_{S^1} dx \left[ -N \gamma^{-1/2} Y^{A} P_{A} + \frac{1}{2} |g|^{1/2} g^{AB} \Pi_{A} \Pi_{B} + Y^{A,1} \Pi_{A,1} \right] \]
\[ - \Lambda_{A} (X^{A}_{1} - N^{-1} \gamma^{-1/2} \gamma_{11} Y^{A,1} \right]. \] (B26)

Varying $\Lambda_{A}^{1}$ give us the equation
\[ Y^{A,1} = N \gamma^{1/2} \gamma^{11} X^{A}_{1} \] (B27)
which enables us to eliminate both $Y^{A,1}$ and $\Lambda_{A}^{1}$ from the action. Comparison of (B26) with (B20) yields the harmonic energy density
\[ H^{H} = -\gamma^{-1/2} Y^{A} P_{A} + \gamma^{1/2} X^{A,1} \Pi_{A,1} + \frac{1}{2} \gamma^{1/2} g^{AB} \Pi_{A} \Pi_{B}. \] (B28)
Here the metric $g^{AB}$ is to be expressed in terms of the configuration variables $X^{A}$, $Y^{A}$, and $\gamma_{11}:
\[ g^{AB} = |g|^{-1} g_{\alpha\beta} Y^{A\alpha} Y^{B\beta} = -\gamma^{-1} Y^{A} Y^{B} + \gamma^{11} X^{A}_{1} X^{B}_{1}. \] (B29)

The pairs of conjugate variables describing the harmonic fields are $X^{A}$, $P_{A}$ and $Y^{A}$, $\Pi_{A}$. With respect to $\text{Diff}S^1$, $X^{A}$ and $\Pi_{A}$ behave as scalars, and $Y^{A}$ and $P_{A}$ as scalar densities. The momentum density $H_{1}^{H}$ must generate the changes of these variables under $L\text{Diff}S^1$. This requirement dictates the functional form of $H_{1}^{H}$ (Dirac 1951, Kuchař 1976a (section 5)) as follows:
\[ H_{1}^{H} = X^{A}_{1} P_{A} - Y^{A} \Pi_{A,1}. \] (B30)
The energy density (B28) and the momentum density (B30) complete the string expressions (B4) and (B5) into the super-Hamiltonian (B21) and the supermomentum (B22) of harmonic string theory.
Diffeomorphisms and harmonic coordinates

B.4. Representing LDiffm and recovering the standard string theory

As in section 4, the constraints (B21) and (B22) are linear in the momentum $P_A$ conjugate to the embedding:

$$H = n^A P_A + \bar{H} = \gamma^{1/2} X^{A1} \Pi_{A1} + \frac{1}{2} \gamma^{1/2} g^{AB} \Pi_A \Pi_B + H^S$$

$$H = X^A P_A + \bar{H}_1 = -Y^A \Pi_{A1} + H^S_1. \quad (B31)$$

This linearity of the projected constraints (B31) enables us to replace them by the unprojected constraints $H_A$, where

$$H_A(x) := P_A(x) - \bar{H}(x; X^B, Y^B, \Pi_B, \gamma_{11}) n_A(x; X^B, Y^B)$$

$$+ \bar{H}_1(x; Y^B, \Pi_B) X^{A1}_A(x; X^B, Y^B). \quad (B32)$$

In the constraints (B32), $P_A(x)$ is cleanly separated from the rest of the canonical variables, and as a consequence of this the Poisson brackets between them vanish strongly, according to the argument given in section 4:

$$\{H_A(x), H_B(x')\} = 0. \quad (B33)$$

Equation (B33) and the separation of $P_A(x)$ from the rest of the canonical variables in (B32) allow us to construct the Isham–Kuchař representation of spacetime diffeomorphisms as usual: Let the (complete) world sheet vector fields $U$ and $V$ on $m$ be two generators of the world sheet diffeomorphism group. Restrict these fields to the embeddings, so that we have $U(x; X^A) := U(X^A(x))$ and $V(x; X^A) := V(X^A(x))$. Form the total Hamiltonian $H(U)$ by the mapping

$$U \longmapsto H(U) := \int_{S^1} dx \ U^A(X^B(x)) H_A(x; \gamma_{11}, X^A, P_A, Y^A, \Pi_A) \quad (B34)$$

and similarly for $V$ and $H(V)$. Then, as desired,

$$\{H(U), H(V)\} = H([UV]). \quad (B35)$$

To make our theory coincide with the standard string theory, we must impose the constraint

$$C_1 := \Pi_A \approx 0 \quad (B36)$$

so as to ensure that the $X^A$ coordinates satisfy the harmonic conditions. Then, in order that these conditions hold throughout the evolution generated by the total Hamiltonian $H(N, N^1)$ given by (B4), (B5), (B20) and (B31), we require that

$$\dot{C}_1 = \{C_1, H(N, N^1)\} \approx 0 \quad (B37)$$

which leads, by use of (B36), to the secondary constraint

$$C_2 := P_A \approx 0. \quad (B38)$$
From (B36) and (B38) we see that $H$ and $H_1$ as given by (B31) reduce to the usual super-Hamiltonian and supermomentum constraints $H^S$ and $H^S_1$ of string theory, as given by (B4) and (B5). Furthermore, it is easy to show that the supplementary constraints $C_1$ and $C_2$ are both maintained into the future through the dynamics generated by the total Hamiltonian $H(N, N^1)$, since

$$\dot{C}_1 = \{C_1, H(N, N^1)\} \approx 0 \approx \{C_2, H(N, N^1)\} = \dot{C}_2.$$  \hfill (B39)

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