Verifying Identities without Begging the Question

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Abstract

The best way to verify a new identity is to begin with one side of the equation and show that by using known identities it can be transformed step by step into the other side.

A popular alternative used by many students, and countenanced even in some textbooks, is to work downwards through two columns having an equals sign – typically surmounted by a small question mark – positioned between them. I contend that this second approach is inferior to the procedure described in the first paragraph, as it may cause one to come perilously close to committing the logical fallacies of petitio principii and affirming the consequent.

Introduction

In this article I shall explain my misgivings concerning a certain technique of verifying identities. Much as I am troubled by the method in question, it is in widespread use amongst high school pupils and university students, and is practised even by some textbook authors. Consequently, I sometimes wonder if my dislike of this approach is a mere prejudice, or if it is indeed a justifiable response to real logical error. Once I have argued my case, I hope to find out whether other mathematics teachers believe my objection to be no more than an idiosyncratic personal aversion, or an important point that we all ought to be emphasizing in our classrooms.
Two Ways of Verifying Identities

Since even an elementary example will suffice to illustrate the contrast between the different strategies of proof, let us verify the trigonometric identity \( \cos^2 x + \tan^2 x \cos^2 x = 1 \).

**Method 1** (which I advocate)

Because the left-hand side (LHS) of this equation is the more complicated of the two sides, we begin with it and show that it can be transformed into the right-hand side (RHS):

\[
\text{LHS} = \cos^2 x + \tan^2 x \cos^2 x = \cos^2 x + \frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x
\]

\[
= \cos^2 x + \sin^2 x = 1 = \text{RHS}\quad \text{Q.E.D.}
\]

**Method 2**

In the (non-honours) pre-calculus textbook\(^2\) we used until 2008 at the high school where I teach, the verification of this identity is presented as follows:

\[
\cos^2 x + \tan^2 x \cos^2 x \quad ? \quad 1
\]

\[
\cos^2 x + \frac{\sin^2 x}{\cos^2 x} \cdot \cos^2 x \quad ? \quad 1
\]

\[
\cos^2 x + \sin^2 x \quad ? \quad 1
\]

\[
1 = 1
\]

**Objections to the Second Method**

Method 2 is similar to Method 1 in that the LHS is gradually transformed into the RHS through the use of known identities, so perhaps there is no *fundamental* objection to it. However, because the LHS is shown as being (at least provisionally) equal to the RHS at each step, I consider this approach somewhat dubious, as it seems to come uncomfortably close to ‘begging the question’. To beg the question is tacitly or surreptitiously to *assume* the truth of the very proposition one is supposed to be proving. With apologies to Epimenides (and St Paul in Titus
1.12), an example would be to assert that ‘No one from Crete will ever tell the truth. We can be sure of this because Cretans always lie.’ Here the premiss and the conclusion of the argument, although formulated in different words, are in reality the same, which means that the premiss cannot be relevant to proving the conclusion. This variety of circular reasoning is also known as *petitio principii* (that is, simply laying claim to a principle, or taking it for granted).

Notice that in Method 2 the last line of the proof is ‘1 = 1’, the implied claim seeming to be that this result is the climax of our investigation and that it establishes the truth of the identity. Surely, though, 1 = 1 is no more than a triviality. We always knew it to be the case, and it provides no help at all in proving the identity. To treat 1 = 1 as if it were the crowning achievement of the verification process is, in essence, to fall prey to the fallacy of ‘affirming the consequent’, which is an invalid syllogism of the following form:

**Premisses:**
1. If p, then q (i.e., p implies q).
2. q is true.

**Conclusion:**
Therefore, p is true.

Here is an example of a specious argument of this type in everyday language:

If Charles Dickens wrote *Hamlet*, then he was a great writer.
Dickens was a great writer.
Therefore, Dickens wrote *Hamlet*.

Method 2 would appear to follow this same unhappy pattern, with p being the trigonometric identity to be proved and q the statement that 1 = 1. That is, on the face of it, Method 2 seems to assert that the putative identity is true simply because 1 = 1, which is patent nonsense, since by means of such flawed reasoning one could ‘prove’ absolutely anything, no matter how absurd.

To illustrate further the insidious nature of Method 2, consider an example similar to the one above (in that it still ends with 1 = 1), but in which the original supposed identity is glaringly false:

\[-1 \neq 1\]

Square each side to get:

\[1 = 1\]
No one would be convinced by such a spurious ‘proof’ that \(-1 = 1\), yet the argument employed is superficially close enough to that of Method 2 (admittedly, they are not in fact the same) that it might mislead a student who was not paying careful attention.

I can imagine two possible, if rather feeble, attempts to defend Method 2. In the first instance, it could perhaps be argued that copying down the (static) RHS of the identity at each vertical step of the procedure helps the student stay clearly focused on the ultimate goal of the proof. In my opinion, however, this is an exceedingly meagre profit to reap from such a large investment in added length and questionable logic.

A second apologist might offer something along the following lines: ‘Once axioms and definitions have been laid down, would not every identity be immediately apparent to a perfect intelligence? From the perspective of such an unlimited mind, all identities would be no more than blatant tautologies, on a par with something as self-evident as \(1 = 1\). Hence, any objection to the use of that equation in our proof is not so much logical as psychological.’ My answer to such a proposed defence of Method 2 is that when we verify an identity we are endeavouring to prove it not to an infinite intelligence but to our fellow human beings, for whom many identities are far from obvious. In order, therefore, to demonstrate convincingly the truth of an identity, we need to follow a gradual, step-by-step process, not simply declare that it is ‘clearly’ true, and we must be especially careful not to claim – or even subtly to suggest – that its truth follows from the reflexive property of equality alone: e.g., in this case, the mere fact that \(1 = 1\).

**Summary**

To avoid so much as the appearance of begging the question when verifying an identity, one ought to treat each side of the equation *separately* from the other until the truth of the identity is established, as is done in Method 1. The root cause of trouble in Method 2 is readily traced to the equals sign with a question mark above it, as well as to the distracting presence of the right-
hand column. All the transformations occur in the left-hand column only, as progress is made downwards, and the successive vertical steps of Method 2 are in fact identical to the horizontal steps of Method 1. The equals sign and the (inert) right-hand column of Method 2 are thus revealed to be quite superfluous – contributing nothing except additional length and a potential source of logical confusion – so what good reason can there be for retaining them? We are well advised simply to discard them and to use Method 1 instead.

**Conclusion**

I believe that the logic of Method 1 is unassailable, whereas (in the light of the foregoing considerations) that of Method 2 is at best suspect. Therefore, we ought both to advocate Method 1 and to point out the weaknesses of Method 2 to our students. I should welcome a discussion or critique of my position, as it may, after all, be merely a storm in a teacup – much ado about very little. Perhaps Method 2 is only a pet hate of mine and is in fact defensible on logical or pedagogical grounds. What do you think?

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