1 Introduction

This note presents a Alt-Alz telescope tracking model. The tracking model is developed in order to relate actual elevation and azimuth of a celestial object tracked by the telescope to the "raw" azimuth and elevation angle by which the telescope mount is turned around its two axis. So as a starting point, the tracking model takes into account the fact that the telescope points to actual azimuth $\phi_0$ and elevation $\theta_0$ when the raw telescope mount azimuth and elevation angles are both set to zero. Then, the tracking model accounts for the imperfect verticality of the azimuth axis. The azimuth axis makes an angle $\alpha > 0$ with the vertical in the direction of azimuth $\psi$ (The azimuth angle is $0^\circ$ due north, $90^\circ$ due east, $180^\circ$ due south and $270^\circ$ due west). The tracking model also accounts for the non perpendicularity of the azimuth and elevation axis. The angle between the elevation and azimuth axis deviates from perpendicularity by an angle $\delta$.

2 From raw to real coordinates

In order to establish the sky direction $(\theta, \phi)$ achieved for a given set of raw telescope mount angle $(\theta_R, \phi_R)$ we start from the situation in which the raw angles are both equal to zero and the telescope is pointing in the direction $\phi_0, \theta_0$. Then we proceed tieh the rotation by $\theta_R$ and $\phi_R$ around the elevation and rotation axis in one or the other order as the two operations commute.

2.1 Vectors

To a given set of azimuth and elevation $(\phi, \theta)$ we associate a vector $u = (x, y, z)$ in a coordinate system where $x$ increases horizontally to the East, $y$ increases horizontally to the north and $z$ increases vertically.

$$u_0 = \begin{pmatrix} cos(\theta)sin(\phi) \\ cos(\theta)cos(\phi) \\ sin(\theta) \end{pmatrix}$$
\[ x = \cos(\theta) \sin(\phi) \]
\[ y = \cos(\theta) \cos(\phi) \]
\[ z = \sin(\theta) \]

2.2 Rotation around the azimuth axis

We start with the rotation around the azimuth axis which, as we will see, also intervenes in the writing of the rotation elevation axis in order to simplify its form. The problem of the azimuth axis rotation is presented on Figure 1.

Figure 1: The coordinate system used to describe rotations around the azimuth axis.

With the telescope pointing along a direction of components \( (x, y, z) \), the first task at hand is to convert to the \( (x', y', z') \) in which the rotation around the azimuth axis by an angle \( \phi_R \) takes a simple form. This is achieved by a rotation of \(-\psi\) in the \( (x, y) \) plane and then by a rotation of \(-\alpha\) in the \( (y, z) \) plane. Then after completion of this rotation we transform the vector components back to the \( (x, y, z) \) system. This gives the following transformation. Note that the azimuth rotation is a rotation by \(-\phi_R\) in the \( (x', y') \) plane. This corresponds to the fact a positive azimuth is an indirect angle.

\[
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix}
= \begin{pmatrix}
  1 & 0 & 0 \\
  0 & \cos(\alpha) & \sin(\alpha) \\
  0 & -\sin(\alpha) & \cos(\alpha)
\end{pmatrix}
\begin{pmatrix}
  \cos(\psi) & \sin(\psi) & 0 \\
  -\sin(\psi) & \cos(\psi) & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x \\
  y \\
  z
\end{pmatrix}
\]

\[
\begin{pmatrix}
  x'_1 \\
  y'_1 \\
  z'_1
\end{pmatrix}
= \begin{pmatrix}
  \cos(\phi_R) & \sin(\phi_R) & 0 \\
  -\sin(\phi_R) & \cos(\phi_R) & 0 \\
  0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
  x' \\
  y' \\
  z'
\end{pmatrix}
\]
point along the direction $u$ plane. Noting the elevation axis rotation by $x,y$ telescope in a configuration in which its elevation axis was parallel to the $(A_0,\phi_0)$. We need to get back to the $(x,y,z)$ plane. The elevation rotation is then a rotation by $	heta$.

In what follow, we will note this transformation $A[\phi_R] = (\psi)^{-1}(\alpha)^{-1}(\phi_R)^{-1}(\alpha)(\psi)$. Since this it will be used later on, we note that

$$A[\phi_1]A[\phi_2] = (\psi)^{-1}(\alpha)^{-1}(\phi_1)^{-1}(\phi_2)^{-1}(\alpha)(\psi) = A[\phi_1 + \phi_2]$$

### 2.2.1 Rotation around the elevation axis

The rotation around the elevation axis takes a simple form when the elevation axis is along the $x$ direction. At the time of the rotation around the elevation axis, the raw azimuth may be $\phi_R$ so we start by applying the transformation $A[-\phi_R]$. This does not bring the elevation axis in the $(x,y)$ plane, even when $\delta \neq 0$. We need to apply a rotation around the azimuth axis by $\hat{\psi}$ to bring the elevation axis to be parallel to the $(x,y)$ plane. It is clear that $\psi \neq \hat{\psi}$ since $\psi$ is measured in the plane $(x,y)$ while $\hat{\psi}$ is measured in the $(x',y')$. So we apply an azimuth axis rotation $A[\hat{\psi}]$. The we change coordinate system from $(x,y,z)$ to $(x',y',z')$. Before proceeding with the elevation axis rotation, we still need to bring the elevation axis along the $(x)$ direction by a rotation by $\delta$ in the $(x,z)$ plane. The elevation rotation is then a rotation by $\theta_R$ in the $(y,z)$ plane. After this we need to get back to the $(x,y,z)$ coordinate system and apply $A[-\hat{\psi}]$ and $A[-\phi_R]$ transformations in order to undo the rotation we did to bring the telescope in a configuration in which its elevation axis was parallel to the $(x,y)$ plane. Noting the elevation axis rotation by $\theta_R$, $E[\theta_R]$ and using the same type of notations as previously introduced for the elementary rotations, we obtain the following expression.

$$E[\theta_R] = A[\phi_R]A^{-1}[\hat{\psi}](\psi)^{-1}(\alpha)^{-1}(\delta)^{-1}(\theta_R)(\delta)(\alpha)(\hat{\psi})A[\hat{\psi}]A^{-1}[\phi_R]$$

or

$$E[\theta_R] = A[\phi_R - \hat{\psi}](\psi)^{-1}(\alpha)^{-1}(\delta)^{-1}(\theta_R)(\delta)(\alpha)(\hat{\psi})A[\hat{\psi} - \phi_R]$$

and, replacing $A$ by its expression and simplifying

$$E[\theta_R] = (\psi)^{-1}(\alpha)^{-1}(\phi_R - \hat{\psi})^{-1}(\delta)^{-1}(\theta_R)(\delta)(\phi_R - \hat{\psi})(\alpha)(\psi)$$

### 2.2.2 Combined rotations

We start from the telescope position $\phi_{R_0} = 0$ and $\theta_{R_0} = 0$ in which the telescope point along the direction $u$

$$u_0 = \begin{pmatrix} \cos(\theta_0)\sin(\phi_0) \\ \cos(\theta_0)\cos(\phi_0) \\ \sin(\theta_0) \end{pmatrix}$$
We may then apply the transformation $E[\theta_R]A[\phi_R]$ or $A[\phi_R]E[\theta_R]$ which should be equivalent as we verify here.

After simplification of canceling rotations in the coordinate transformations

$$E[\theta_R]A[\phi_R] = (\psi)^{-1}(\alpha)^{-1}(\phi_R - \hat{\psi})^{-1}(\delta)^{-1}(\theta_R)(\phi_R - \hat{\psi})(\phi_R)^{-1}(\alpha)(\psi)$$

and

$$E[\theta_R]A[\phi_R] = (\psi)^{-1}(\alpha)^{-1}(\phi_R)^{-1}(\hat{\psi})(\delta)^{-1}(\theta_R)(\hat{\psi})^{-1}(\alpha)(\psi)$$

If, on the contrary we start with the rotation around the elevation axis,

$$E[\theta_R](\psi)^{-1}(\alpha)^{-1}(\phi_R - \hat{\psi})^{-1}(\delta)^{-1}(\theta_R)(\phi_R - \hat{\psi})(\alpha)(\psi)$$

and, given the values of $\phi_{R_0}$ and $\theta_{R_0}$

$$E[\theta_R] = (\psi)^{-1}(\alpha)^{-1}(\phi_R - \hat{\psi})^{-1}(\theta_R)(\phi_R - \hat{\psi})(\alpha)(\psi)$$

so, combining the two rotation,

$$A[\phi_R]E[\theta_R] = (\psi)^{-1}(\alpha)^{-1}(\phi_R)^{-1}(\hat{\psi})(\delta)^{-1}(\theta_R)(\hat{\psi})^{-1}(\alpha)(\psi)$$

and we verify that $A[\phi_R]E[\theta_R] = E[\theta_R]A[\phi_R]$

3 Optimization of the parameters

The c code for the transformation derived above is given in the next section. In order to establish the tracking model parameters $\psi, \alpha, \delta, \psi_0, \theta_0$, we have made series of star pointing with both telescopes. For each star $(\phi, \theta)$ we entered $\phi$ and $\theta$ as the raw azimuth and elevation. We then applied corrections $\Delta \phi_R$ and $\Delta \theta_R$ to the raw azimuth and elevation so as to bring the star on the optical axis of the telescope. Ideally this should ensure $\tilde{u} = A[\phi + \Delta \phi_R]E[\theta + \Delta \theta_R]u_0 = u$. For each star $i$ we calculate the angle $\epsilon_i$ between $u$ and $\tilde{u}$ and compute $\chi^2 = \sum \epsilon_i^2$. This $\chi^2$ is then minimized with respect to $\{\psi, \alpha, \delta, \psi_0, \theta_0\}$.

The result obtained from a first set of data is indicated for both telescopes in the following tables.
Table 1: East telescope with $\psi = 128.16^\circ$, $\alpha = 0.308^\circ$, $\delta = 0.12^\circ$, $\phi_0 = 0.032^\circ$ and $\theta_0 = 0.112^\circ$, the standard error is 0.027°

<table>
<thead>
<tr>
<th>Star</th>
<th>$\phi$</th>
<th>$\theta$</th>
<th>$\Delta \phi_R$</th>
<th>$\Delta \theta_R$</th>
<th>$\phi$</th>
<th>$\theta$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>00.418</td>
<td>40.140</td>
<td>-0.350</td>
<td>-0.130</td>
<td>0.415</td>
<td>40.121</td>
<td>0.019</td>
</tr>
<tr>
<td>2</td>
<td>40.370</td>
<td>32.470</td>
<td>-0.300</td>
<td>0.100</td>
<td>40.380</td>
<td>32.480</td>
<td>0.014</td>
</tr>
<tr>
<td>3</td>
<td>81.190</td>
<td>36.030</td>
<td>-0.300</td>
<td>0.290</td>
<td>81.182</td>
<td>36.032</td>
<td>0.006</td>
</tr>
<tr>
<td>4</td>
<td>66.570</td>
<td>69.390</td>
<td>-1.170</td>
<td>0.190</td>
<td>66.537</td>
<td>69.357</td>
<td>0.035</td>
</tr>
<tr>
<td>5</td>
<td>155.660</td>
<td>56.440</td>
<td>-0.040</td>
<td>0.360</td>
<td>155.645</td>
<td>56.449</td>
<td>0.012</td>
</tr>
<tr>
<td>6</td>
<td>214.510</td>
<td>14.380</td>
<td>-0.010</td>
<td>0.130</td>
<td>214.485</td>
<td>14.412</td>
<td>0.040</td>
</tr>
<tr>
<td>7</td>
<td>182.740</td>
<td>23.290</td>
<td>0.010</td>
<td>0.280</td>
<td>182.728</td>
<td>23.313</td>
<td>0.026</td>
</tr>
<tr>
<td>8</td>
<td>270.310</td>
<td>43.070</td>
<td>0.050</td>
<td>-0.180</td>
<td>270.340</td>
<td>43.055</td>
<td>0.026</td>
</tr>
<tr>
<td>9</td>
<td>215.140</td>
<td>26.170</td>
<td>0.050</td>
<td>0.100</td>
<td>215.134</td>
<td>26.176</td>
<td>0.008</td>
</tr>
<tr>
<td>10</td>
<td>228.490</td>
<td>53.040</td>
<td>0.200</td>
<td>0.000</td>
<td>228.501</td>
<td>53.018</td>
<td>0.023</td>
</tr>
<tr>
<td>11</td>
<td>264.440</td>
<td>79.200</td>
<td>0.130</td>
<td>-0.190</td>
<td>264.258</td>
<td>79.155</td>
<td>0.057</td>
</tr>
<tr>
<td>12</td>
<td>310.250</td>
<td>30.980</td>
<td>-0.100</td>
<td>-0.220</td>
<td>310.265</td>
<td>30.990</td>
<td>0.016</td>
</tr>
<tr>
<td>13</td>
<td>318.350</td>
<td>29.820</td>
<td>-0.120</td>
<td>-0.220</td>
<td>318.366</td>
<td>29.825</td>
<td>0.015</td>
</tr>
<tr>
<td>14</td>
<td>322.360</td>
<td>26.070</td>
<td>-0.120</td>
<td>-0.220</td>
<td>322.370</td>
<td>26.070</td>
<td>0.009</td>
</tr>
<tr>
<td>15</td>
<td>337.890</td>
<td>19.280</td>
<td>-0.150</td>
<td>-0.160</td>
<td>337.868</td>
<td>19.309</td>
<td>0.036</td>
</tr>
<tr>
<td>16</td>
<td>335.280</td>
<td>14.500</td>
<td>-0.010</td>
<td>-0.190</td>
<td>335.258</td>
<td>14.536</td>
<td>0.036</td>
</tr>
<tr>
<td>17</td>
<td>335.280</td>
<td>79.820</td>
<td>-0.100</td>
<td>-0.210</td>
<td>335.265</td>
<td>79.890</td>
<td>0.016</td>
</tr>
</tbody>
</table>

Table 2: West telescope with $\psi = 162.72^\circ$, $\alpha = 0.21^\circ$, $\delta = 0.56^\circ$, $\phi_0 = -0.72^\circ$ and $\theta_0 = 0.248^\circ$, the standard error is 0.059°

<table>
<thead>
<tr>
<th>Star</th>
<th>$\phi$</th>
<th>$\theta$</th>
<th>$\Delta \phi_R$</th>
<th>$\Delta \theta_R$</th>
<th>$\phi$</th>
<th>$\theta$</th>
<th>$\epsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>180.700</td>
<td>23.010</td>
<td>0.500</td>
<td>0.110</td>
<td>180.628</td>
<td>22.953</td>
<td>0.088</td>
</tr>
<tr>
<td>2</td>
<td>232.400</td>
<td>21.730</td>
<td>0.610</td>
<td>-0.010</td>
<td>232.377</td>
<td>21.684</td>
<td>0.051</td>
</tr>
<tr>
<td>3</td>
<td>324.170</td>
<td>36.960</td>
<td>0.540</td>
<td>-0.220</td>
<td>324.178</td>
<td>36.982</td>
<td>0.023</td>
</tr>
<tr>
<td>4</td>
<td>00.810</td>
<td>39.840</td>
<td>0.320</td>
<td>-0.220</td>
<td>0.714</td>
<td>39.863</td>
<td>0.077</td>
</tr>
<tr>
<td>5</td>
<td>64.380</td>
<td>49.970</td>
<td>0.140</td>
<td>-0.070</td>
<td>64.321</td>
<td>49.968</td>
<td>0.038</td>
</tr>
<tr>
<td>6</td>
<td>88.270</td>
<td>72.160</td>
<td>-0.100</td>
<td>0.050</td>
<td>88.206</td>
<td>72.188</td>
<td>0.034</td>
</tr>
<tr>
<td>7</td>
<td>120.450</td>
<td>42.250</td>
<td>0.300</td>
<td>0.090</td>
<td>120.418</td>
<td>42.217</td>
<td>0.040</td>
</tr>
<tr>
<td>8</td>
<td>255.170</td>
<td>45.960</td>
<td>0.750</td>
<td>-0.100</td>
<td>255.240</td>
<td>45.906</td>
<td>0.072</td>
</tr>
<tr>
<td>9</td>
<td>302.330</td>
<td>51.390</td>
<td>0.700</td>
<td>-0.200</td>
<td>302.402</td>
<td>51.391</td>
<td>0.045</td>
</tr>
<tr>
<td>10</td>
<td>311.800</td>
<td>47.590</td>
<td>0.660</td>
<td>-0.200</td>
<td>311.884</td>
<td>47.612</td>
<td>0.061</td>
</tr>
<tr>
<td>11</td>
<td>314.350</td>
<td>42.990</td>
<td>0.650</td>
<td>-0.220</td>
<td>314.442</td>
<td>42.997</td>
<td>0.067</td>
</tr>
<tr>
<td>12</td>
<td>315.380</td>
<td>33.380</td>
<td>0.640</td>
<td>-0.210</td>
<td>315.460</td>
<td>33.400</td>
<td>0.070</td>
</tr>
<tr>
<td>13</td>
<td>322.540</td>
<td>27.620</td>
<td>0.630</td>
<td>-0.210</td>
<td>322.610</td>
<td>27.651</td>
<td>0.069</td>
</tr>
<tr>
<td>14</td>
<td>37.640</td>
<td>27.110</td>
<td>0.370</td>
<td>-0.100</td>
<td>37.577</td>
<td>27.173</td>
<td>0.084</td>
</tr>
<tr>
<td>15</td>
<td>102.200</td>
<td>83.100</td>
<td>-0.570</td>
<td>0.096</td>
<td>101.825</td>
<td>83.126</td>
<td>0.052</td>
</tr>
<tr>
<td>16</td>
<td>203.680</td>
<td>19.250</td>
<td>0.620</td>
<td>0.090</td>
<td>203.684</td>
<td>19.216</td>
<td>0.034</td>
</tr>
</tbody>
</table>
4 C implementation of the $A[\phi_R]E[\theta_R]$ transformation

The input variable are $\phi = \psi, \alpha, \phi_0, \theta_0, \delta, \phi_R$ and $\theta_R$. The output variables are $\phi = \phi$ and $\theta = \theta$ the actual azimuth and elevation after completion of the rotations by $\phi_R$ and $\theta_R$. The variable $r \phi = \psi$ and the function makes the approximation $\psi = \psi$. DEGTORAD is the number of degrees in one radian.

```c
int model(double phi, double tlt, double offaz, double offel, double crook,
          double raz, double rel, double *taz, double *tel)
    phi=phi/DEGTORAD; //Azimuth of the azimuth axis tilt
    tlt=tlt/DEGTORAD; //Azimuth axis tilt
    crook=crook/DEGTORAD; //Deviation from 90 of the angle between mount
    axis rel=rel/DEGTORAD; //Raw elevation
    raz=raz/DEGTORAD; //Raw azimuth
    offaz=offaz/DEGTORAD; //Actual Azimuth when raw az=0 and raw el=0
    offel=offel/DEGTORAD; //Actual elevation when raw az=0 and raw el=0
    double a11;
    double a12;
    double a13;
    double a21;
    double a22;
    double a23;
    double a31;
    double a32;
    double a33;
    double xt;
    double yt;
    double zt;
    //Function making the approximation psi=psi. DEGTORAD is the number of degrees
    //in one radian.
    double x=cos(offel)*sin(offaz);
    double y=cos(offel)*cos(offaz);
    double z=sin(offel);
    //double x=0;
    //double y=1;
    //double z=0;
    //raz=raz+offaz;
    //rel=rel+offel;
```
// Rotation by phi
a11 = cos(phi);
a12 = -sin(phi);
a13 = 0;
a21 = sin(phi);
a22 = cos(phi);
a23 = 0;
a31 = 0;
a32 = 0;
a33 = 1;
xt = a11*x + a12*y + a13*z;
yt = a21*x + a22*y + a23*z;
zt = a31*x + a32*y + a33*z;
x = xt;
y = yt;
z = zt;

// Rotation by tilt
a11 = 1;
a12 = 0;
a13 = 0;
a21 = 0;
a22 = cos(tlt);
a23 = -sin(tlt);
a31 = 0;
a32 = sin(tlt);
a33 = cos(tlt);
xt = a11*x + a12*y + a13*z;
yt = a21*x + a22*y + a23*z;
zt = a31*x + a32*y + a33*z;
x = xt;
y = yt;
z = zt;

double rphi = phi;  // This is an approximation which I will look into

// Rotation by rphi
a11 = cos(rphi);
a12 = sin(rphi);
a13 = 0;
a21 = -sin(rphi);
a22 = cos(rphi);
a23 = 0;
a31 = 0;
a32 = 0;
a33 = 1;
xt = a11*x + a12*y + a13*z;
yt = \(a_{21}x + a_{22}y + a_{23}z\);
zt = \(a_{31}x + a_{32}y + a_{33}z\);
x = xt;
y = yt;
z = zt;

// Rotation by the crook
a_{11} = \cos(crook);
a_{12} = 0;
a_{13} = \sin(crook);
a_{21} = 0;
a_{22} = 1;
a_{23} = 0;
a_{31} = 
\sin(crook);
a_{32} = 0;
a_{33} = \cos(crook);
xt = a_{11}x + a_{12}y + a_{13}z;
yt = a_{21}x + a_{22}y + a_{23}z;
zt = a_{31}x + a_{32}y + a_{33}z;
x = xt;
y = yt;
z = zt;

// Rotation by the elevation
a_{11} = 1;
a_{12} = 0;
a_{13} = 0;
a_{21} = 0;
a_{22} = \cos(rel);
a_{23} = -\sin(rel);
a_{31} = 0;
a_{32} = \sin(rel);
a_{33} = \cos(rel);
xt = a_{11}x + a_{12}y + a_{13}z;
yt = a_{21}x + a_{22}y + a_{23}z;
zt = a_{31}x + a_{32}y + a_{33}z;
x = xt;
y = yt;
z = zt;

// Rotation by -crook
a_{11} = \cos(crook);
a_{12} = 0;
a_{13} = \sin(crook);
a_{21} = 0;
a_{22} = 1;
a23=0;
a31=-\sin(\text{crook}) ;
a32=0 ;
a33=\cos(\text{crook}) ;
xt=a11*x+a12*y+a13*z ;
yt=a21*x+a22*y+a23*z ;
zt=a31*x+a32*y+a33*z ;
x=xt ;
y= yt ;
z= zt ;

//Rotation by rphi
a11=\cos(\text{rphi}) ;
a12=-\sin(\text{rphi}) ;
a13=0 ;
a21=\sin(\text{rphi}) ;
a22=\cos(\text{rphi}) ;
a23=0 ;
a31=0 ;
a32=0 ;
a33=1 ;
xt=a11*x+a12*y+a13*z ;
yt=a21*x+a22*y+a23*z ;
zt=a31*x+a32*y+a33*z ;
x=xt ;
y= yt ;
z= zt ;

//Rotation by the -raw azimuth (negative sign due to azimuth definition)
a11=\cos(\text{raz}) ;
a12=\sin(\text{raz}) ;
a13=0 ;
a21=-\sin(\text{raz}) ;
a22=\cos(\text{raz}) ;
a23=0 ;
a31=0 ;
a32=0 ;
a33=1 ;
xt=a11*x+a12*y+a13*z ;
yt=a21*x+a22*y+a23*z ;
zt=a31*x+a32*y+a33*z ;
x=xt ;
y= yt ;
z= zt ;

//Rotation by tilt
a11=1;  
a12=0;  
a13=0;  
a21=0;  
a22=cos(tlt);  
a23=sin(tlt);  
a31=0;  
a32=-sin(tlt);  
a33=cos(tlt);  
xt=a11*x+a12*y+a13*z;  
yt=a21*x+a22*y+a23*z;  
zt=a31*x+a32*y+a33*z;  
x=xt;  
y=yt;  
z=zt;  

//Rotation by phi  
a11=cos(phi);  
a12=sin(phi);  
a13=0;  
a21=-sin(phi);  
a22=cos(phi);  
a23=0;  
a31=0;  
a32=0;  
a33=1;  
xt=a11*x+a12*y+a13*z;  
yt=a21*x+a22*y+a23*z;  
zt=a31*x+a32*y+a33*z;  
x=xt;  
y=yt;  
z=zt;  

tel=asin(z);  
taz=atan2(x,y);  
if(*taz<0) *taz=*taz+2*PI;  

tel=*tel*DEGTORAD;  
taz=*taz*DEGTORAD;  

return 0;  

//======================================================  
//======================================================