Lecture 10: Propagation of Uncertainties

"Uncertainty"

Photograph by Michelle Lane

Physics 3719
Spring Semester 2011
Note on Computers in Lab:

- Computers in SP 306/307 will have the login procedure changed. (Mandate for all campus computers.)
- Use same login (UID and password) as you use for the Campus Information System.
- This change is effective today.
Discussion

• Which will give the better uncertainty in the measurement of the rate \( r \) of radioactive decays:
  - Counting the number of decays in 100 seconds?
  - Counting the number of decays in 10 seconds, doing this 10 times, then taking the average?
Lab 3 Experiments

- Gravitational Constant G
- Speed of Light C
- Electron Charge-to-Mass Ratio e/m
- Millikan Oil Drop Experiment
- Photoelectric Effect
- Hydrogen Balmer Spectrum
- Franck-Hertz Experiment
- High-Resolution Optical Spectroscopy
Propagation of Uncertainties
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\[ f = f(a, b, c \ldots) \]

\[ S_f^2 = \left( \frac{\partial f}{\partial a} \right)^2 S_a^2 + \left( \frac{\partial f}{\partial b} \right)^2 S_b^2 + \left( \frac{\partial f}{\partial c} \right)^2 S_c^2 + \ldots \]
Propagation of Uncertainties

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Where is this from?

14 January 2011
• Consider function of one variable \( f(x) \)

• Assume errors small, so \( f(x) \sim \text{linear} \)

\[
S_f = f(x + S_x) - f(x) \approx \frac{\Delta f}{\Delta x} \cdot S_x \rightarrow \frac{\partial f}{\partial x} \cdot S_x
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Generally we express errors as positive, so...

$$S_f = \left| \frac{\partial f}{\partial x} \right| \cdot S_x$$
Now, what about multivariate functions e.g. $f = f(x,y)$?

Assume errors small, so $f(x) \sim$ linear

\[ f = f(x, y) \]

\[ S_f = \left| \frac{\partial f}{\partial x} \right| \cdot S_x + \left| \frac{\partial f}{\partial y} \right| \cdot S_y \]
• e.g. Measure the heights $x$ and $y$ of two people

• What is the uncertainty in the sum of heights $q = x + y$?

$$q = x + y$$

$$S_q = \left| \frac{\partial q}{\partial x} \right| \cdot S_x + \left| \frac{\partial q}{\partial y} \right| \cdot S_y = S_x + S_y$$

$$q = (x + y) \pm (S_x + S_y)$$

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$$q \pm (S_x + S_y)$$
Consider Two Cases:

- $x$ and $y$ have correlated errors (e.g. ruler is off)
- Expression above gives upper bound to errors

- $x$ and $y$ have uncorrelated errors
- Expression above overestimates error

$$q = (x + y) \pm (S_x + S_y)$$
How do we correctly treat functions of two (or more) variables?
Now, in the case where there are two independent variables and $q_i = q(x_i, y_i)$, we have

$$q_i \sim q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})$$

or, by noting that $\bar{q} = q(\bar{x}, \bar{y})$ (proof left as exercise), we get for the RMS of $q$

$$\sigma_q^2 = \frac{1}{N} \sum_{i=1}^{N} (q_i - \bar{q})^2$$

$$\sim \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y}) \right]^2$$

$$\sim \left( \frac{\partial q}{\partial x} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 + \left( \frac{\partial q}{\partial y} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 + \left( \frac{\partial q}{\partial x} \right) \left( \frac{\partial q}{\partial y} \right) \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})$$

$$\sim \left( \frac{\partial q}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial q}{\partial y} \right)^2 \sigma_y^2 + \left( \frac{\partial q}{\partial x} \right) \left( \frac{\partial q}{\partial y} \right) \sigma_{xy}$$

where we have introduced a new term, the covariance of $x$ and $y$:

$$\sigma_{xy} \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})$$
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where we have introduced a new term, the covariance of \( x \) and \( y \):

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\sigma_{xy} \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})
\]
Propagation of Uncertainty (neglecting correlations)

\[ f = a + b \quad \rightarrow \quad \sigma_f = (\sigma_a^2 + \sigma_b^2)^{1/2} \]

\[ f = a - b \quad \rightarrow \quad \sigma_f = (\sigma_a^2 + \sigma_b^2)^{1/2} \]

\[ f = a \times b \quad \rightarrow \quad \frac{\sigma_f}{f} = \left( \left( \frac{\sigma_a}{a} \right)^2 + \left( \frac{\sigma_b}{b} \right)^2 \right)^{1/2} \]

\[ f = a \div b \quad \rightarrow \quad \frac{\sigma_f}{f} = \left( \left( \frac{\sigma_a}{a} \right)^2 + \left( \frac{\sigma_b}{b} \right)^2 \right)^{1/2} \]

\[ f = a^n \quad \rightarrow \quad \frac{\sigma_f}{f} = n \left( \frac{\sigma_a}{a} \right) \]
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Examples

- What if $f = a^n/b$ ?
- What if $f = a + be^c$ ?
EXAMPLE: Suppose we’re given the relation

\[ F = \frac{ABC}{D + E} \]

where \( A, B, C, D, E \) are given by

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<tbody>
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• QUICK: What is uncertainty in \( F \)?
EXAMPLE: Suppose we’re given the relation

$$F = \frac{ABC}{D + E}$$

where $A$, $B$, $C$, $D$, $E$ are given by

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- **QUICK: What is uncertainty in $F$?**
- **$B$ has 10% relative uncertainty**
EXAMPLE: Suppose we’re given the relation

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where \( A, B, C, D, E \) are given by

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\begin{array}{c|c}
A & 28.0 \pm 0.3 \\
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- **QUICK**: What is uncertainty in \( F \)?
- **B** has 10% relative uncertainty
- All others have 1% or less
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\]

- **QUICK:** What is uncertainty in \( F \)?
- \( B \) has 10% relative uncertainty
- All others have 1% or less
- \( F \) has ~ same relative uncertainty as \( B \): \( S_F \sim 15 \)
Covariance:

\[ \sigma_{xy} \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y}) \]
Additional Reading and Problems

• Read in **Taylor**:
  - Ch 3: Propagation of Uncertainties
  - Ch 4: Statistical Analysis of Random Uncertainties
  - Ch 9: Covariance and Correlation

• Try the problems:
  - Prove the relations on slide #15
  - 3.1, 3.2, 3.4, 3.15, 3.24
  - 4.6, 4.8, 4.19, 4.28
  - 9.2, 9.4, 9.8, 9.16