Lab 3 Experiments

- Gravitational Constant G Jensen, Blackmon (AM)
- Speed of Light C Ramirez, Thomas (PM)
- Electron Charge-to-Mass Ratio $e/m$ Brown, Blatter (PM)
- Millikan Oil Drop Experiment Baehr, Martineau (AM)
- Photoelectric Effect Peterson, Ingebretsen (PM)
- Hydrogen Balmer Spectrum Gibbs, Baird (PM)
- Franck-Hertz Experiment Rowley, Bernal (AM)
- High-Res Optical Spectroscopy Richards, Topham (AM)
Now, in the case where there are two independent variables and \( q_i = q(x_i, y_i) \), we have

\[
q_i \sim q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})
\]

or, by noting that \( \bar{q} = q(\bar{x}, \bar{y}) \) (proof left as exercise), we get for the RMS of \( q \)

\[
\sigma_q^2 = \frac{1}{N} \sum_{i=1}^{N} (q_i - \bar{q})^2
\]

\[
\sim \frac{1}{N} \sum_{i=1}^{N} \left[ \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y}) \right]^2
\]

\[
\sim \left( \frac{\partial q}{\partial x} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})^2 + \left( \frac{\partial q}{\partial y} \right)^2 \frac{1}{N} \sum_{i=1}^{N} (y_i - \bar{y})^2 + \left( \frac{\partial q}{\partial x} \right) \left( \frac{\partial q}{\partial y} \right) \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})
\]

\[
\sim \left( \frac{\partial q}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial q}{\partial y} \right)^2 \sigma_y^2 + \left( \frac{\partial q}{\partial x} \right) \left( \frac{\partial q}{\partial y} \right) \sigma_{xy}
\]

where we have introduced a new term, the covariance of \( x \) and \( y \):

\[
\sigma_{xy} \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})
\]
\[
q = q(x, y)
\]

**“Old” Terms**

\[
\sigma_q^2 \sim \left( \frac{\partial q}{\partial x} \right)^2 \sigma_x^2 + \left( \frac{\partial q}{\partial y} \right)^2 \sigma_y^2 + \left( \frac{\partial q}{\partial x} \right) \left( \frac{\partial q}{\partial y} \right) \sigma_{xy}
\]

**New Term**

**“Covariance”**
- form intuitive
- tends to vanish if \( x \) & \( y \) vary independently
- reduces to former result if no correlation
Covariance:

$$\sigma_{xy} \equiv \frac{1}{N} \sum_{i=1}^{N} (x_i - \bar{x})(y_i - \bar{y})$$

![Positive Covariance](image)

![Zero Covariance](image)

![Negative Covariance](image)
Example: Determine the area of a rectangle, with uncertainty, for a series of measurements of its length and width.
Example: Determine the area of a rectangle, with uncertainty, for a series of measurements of its length and width.

\[ A = L \times W \]

\[ \frac{\partial A}{\partial L} = W \]

\[ \frac{\partial A}{\partial W} = L \]

\[ \sigma_A^2 = W^2 \sigma_L^2 + L^2 \sigma_W^2 + 2LW \sigma_{LW} \]
Uncorrelated Errors

- $\sigma_L = 1.96$
- $\sigma_W = 1.25$
- $\sigma_{LW} = 0.14$
- $\sigma_A$ (calculated) = 21.3
- $\sigma_A$ (w/o $\sigma_{LW}$) = 20.8
- $\sigma_A$ (direct) = 20.6
Correlated Errors

- $\sigma_L = 1.78$
- $\sigma_W = 1.25$
- $\sigma_{LW} = 1.67$
- $\sigma_A$ (calculated) = 25.2
- $\sigma_A$ (w/o $\sigma_{LW}$) = 19.2
- $\sigma_A$ (direct) = 25.3
Correlated Errors

- $\sigma_L = 1.78$
- $\sigma_W = 1.25$
- $\sigma_{LW} = 1.67$
- $\sigma_A$ (calculated) = 25.2
- $\sigma_A$ (w/o $\sigma_{LW}$) = 19.2
- $\sigma_A$ (direct) = 25.3

We need $\sigma_{LW}$ term!
Quantifying Correlation in terms of Covariance

- We want to know if two variables (not uncertainties!) are likely to be linearly related.
- e.g. Shoe size and height

Introduce “coefficient of linear correlation” $r$:

$$r = \frac{\sigma_{xy}}{\sigma_x \sigma_y} = \frac{\sum(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum(x_i - \bar{x})^2} \sqrt{\sum(y_i - \bar{y})^2}}$$

$-1 \leq r \leq 1$

perfect anticorrelation

perfect correlation
Look at points from earlier example:

**Uncorrelated**

$$R = \frac{0.14}{1.96 \times 1.25} = 0.056$$

**Correlated**

$$R = \frac{1.67}{1.78 \times 1.25} = 0.75$$
Significance of $r$
Depends on $N$
points

<table>
<thead>
<tr>
<th>$N$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>100</td>
<td>94</td>
<td>87</td>
<td>81</td>
<td>74</td>
<td>67</td>
<td>59</td>
<td>51</td>
<td>41</td>
<td>29</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>100</td>
<td>90</td>
<td>80</td>
<td>70</td>
<td>60</td>
<td>50</td>
<td>40</td>
<td>30</td>
<td>20</td>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>100</td>
<td>87</td>
<td>75</td>
<td>62</td>
<td>50</td>
<td>39</td>
<td>28</td>
<td>19</td>
<td>10</td>
<td>3.7</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>100</td>
<td>85</td>
<td>70</td>
<td>56</td>
<td>43</td>
<td>31</td>
<td>21</td>
<td>12</td>
<td>5.6</td>
<td>1.4</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>100</td>
<td>83</td>
<td>67</td>
<td>51</td>
<td>37</td>
<td>25</td>
<td>15</td>
<td>8.0</td>
<td>3.1</td>
<td>0.6</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>100</td>
<td>81</td>
<td>63</td>
<td>47</td>
<td>33</td>
<td>21</td>
<td>12</td>
<td>5.3</td>
<td>2.1</td>
<td>0.7</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>100</td>
<td>80</td>
<td>61</td>
<td>43</td>
<td>29</td>
<td>17</td>
<td>8.8</td>
<td>3.6</td>
<td>1.0</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>100</td>
<td>78</td>
<td>58</td>
<td>40</td>
<td>25</td>
<td>14</td>
<td>6.7</td>
<td>2.4</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>100</td>
<td>77</td>
<td>55</td>
<td>37</td>
<td>22</td>
<td>12</td>
<td>5.1</td>
<td>1.6</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>12</td>
<td>100</td>
<td>76</td>
<td>53</td>
<td>34</td>
<td>20</td>
<td>9.8</td>
<td>3.9</td>
<td>1.1</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>13</td>
<td>100</td>
<td>75</td>
<td>51</td>
<td>32</td>
<td>18</td>
<td>8.2</td>
<td>3.6</td>
<td>0.8</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>14</td>
<td>100</td>
<td>73</td>
<td>49</td>
<td>29</td>
<td>15</td>
<td>6.9</td>
<td>2.3</td>
<td>0.5</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15</td>
<td>100</td>
<td>72</td>
<td>47</td>
<td>28</td>
<td>14</td>
<td>5.8</td>
<td>1.8</td>
<td>0.4</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>16</td>
<td>100</td>
<td>71</td>
<td>46</td>
<td>26</td>
<td>12</td>
<td>4.9</td>
<td>1.4</td>
<td>0.3</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>17</td>
<td>100</td>
<td>70</td>
<td>44</td>
<td>24</td>
<td>11</td>
<td>4.1</td>
<td>1.1</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>18</td>
<td>100</td>
<td>69</td>
<td>43</td>
<td>23</td>
<td>10</td>
<td>3.5</td>
<td>0.8</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>19</td>
<td>100</td>
<td>68</td>
<td>41</td>
<td>21</td>
<td>9.0</td>
<td>2.9</td>
<td>0.7</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>20</td>
<td>100</td>
<td>67</td>
<td>40</td>
<td>20</td>
<td>8.1</td>
<td>2.5</td>
<td>0.5</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>21</td>
<td>100</td>
<td>66</td>
<td>34</td>
<td>15</td>
<td>4.8</td>
<td>1.1</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>22</td>
<td>100</td>
<td>65</td>
<td>29</td>
<td>11</td>
<td>2.9</td>
<td>0.5</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>23</td>
<td>100</td>
<td>64</td>
<td>25</td>
<td>8.0</td>
<td>1.7</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>24</td>
<td>100</td>
<td>62</td>
<td>22</td>
<td>16</td>
<td>1.1</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>25</td>
<td>100</td>
<td>61</td>
<td>19</td>
<td>4.5</td>
<td>0.6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$0$</th>
<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
<th>0.3</th>
<th>0.35</th>
<th>0.4</th>
<th>0.45</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>100</td>
<td>73</td>
<td>49</td>
<td>30</td>
<td>15</td>
<td>8.0</td>
<td>3.4</td>
<td>1.3</td>
<td>0.4</td>
</tr>
<tr>
<td>30</td>
<td>100</td>
<td>70</td>
<td>45</td>
<td>35</td>
<td>13</td>
<td>5.4</td>
<td>2.0</td>
<td>0.6</td>
<td>0.2</td>
</tr>
<tr>
<td>40</td>
<td>100</td>
<td>68</td>
<td>41</td>
<td>22</td>
<td>11</td>
<td>3.7</td>
<td>1.2</td>
<td>0.3</td>
<td>0.1</td>
</tr>
<tr>
<td>50</td>
<td>100</td>
<td>66</td>
<td>38</td>
<td>18</td>
<td>7.5</td>
<td>2.5</td>
<td>0.7</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>60</td>
<td>100</td>
<td>64</td>
<td>35</td>
<td>16</td>
<td>5.9</td>
<td>1.7</td>
<td>0.4</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>70</td>
<td>100</td>
<td>62</td>
<td>32</td>
<td>14</td>
<td>4.6</td>
<td>1.2</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>80</td>
<td>100</td>
<td>60</td>
<td>29</td>
<td>12</td>
<td>3.9</td>
<td>0.9</td>
<td>0.2</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>90</td>
<td>100</td>
<td>58</td>
<td>26</td>
<td>10</td>
<td>3.2</td>
<td>0.7</td>
<td>0.1</td>
<td>0.1</td>
<td>0.1</td>
</tr>
<tr>
<td>100</td>
<td>100</td>
<td>56</td>
<td>23</td>
<td>8.0</td>
<td>0.6</td>
<td>0.1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table C. The percentage probability $Proba(|r| \geq r_0)$ that $N$ measurements of two uncorrelated variables give a correlation coefficient with $|r| \geq r_0$ as a function of $N$ and $r_0$. (Blanks indicate probabilities less than 0.05%).
Example

A professor wants to know if telling a lot of jokes in class gets him good student evaluations. Over several years, he records the number of jokes he tells and compares it with the overall performance rating which he is given by the students. His data are collected in the table below:

<table>
<thead>
<tr>
<th>Year</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of Jokes</td>
<td>12</td>
<td>18</td>
<td>33</td>
<td>27</td>
<td>39</td>
</tr>
<tr>
<td>Rating</td>
<td>5.1</td>
<td>4.1</td>
<td>3.8</td>
<td>3.3</td>
<td>3.6</td>
</tr>
</tbody>
</table>

Quantitatively evaluate the correlation between jokes and class evaluations. Should the instructor keep telling jokes, or doesn’t it matter?
Joke/Rating Correlation Analysis

- Mean $x = 25.8$
- Mean $y = 3.98$
Joke/Rating Correlation Analysis

- Mean $x = 25.8$
- Mean $y = 3.98$
- $\sigma_x = 9.7857$
- $\sigma_y = 0.617738$
- $\sigma_{xy} = -4.704$
Joke/Rating Correlation Analysis

• Mean x = 25.8
• Mean y = 3.98
• $\sigma_x = 9.7857$
• $\sigma_y = 0.617738$
• $\sigma_{xy} = -4.704$
• $R = -0.778164$
• $P_{\text{chance}} \sim 15\%$