I. INTRODUCTION

Among the long list of possible charged kaon decays the rare $K_{e4}$ decay branch [K$^{\pm}$→π$^+\pi^-e^\pm\nu_e$(K$_{e4}$)] has received particular attention because it was recognized [1] almost coincidently with the observation of the first event for this decay 40 years ago [2] that it could provide important information on the structure of the weak hadronic currents and also on $\pi\pi$ scattering at low energies. The final state interaction of the two pions was expected to manifest itself in an angular correlation between the decay products, namely an asymmetry of the lepton distribution with respect to the plane formed by the two pion momenta. This asymmetry is directly related to the difference between the s- and p-wave scattering phase. What made this four-body semileptonic decay attractive despite its low branching ratio, which was then predicted to be of order $10^{-5}$ [3], is that the two pions are the only hadrons in the final state. For all other reactions used to study the $\pi\pi$ interaction, e.g. $\pi^- p\rightarrow \pi^- \pi^- n$, there is at least one other hadron present in the final state. Thus experimental studies of the $K_{e4}$ decay seem as the cleanest method to determine the isospin zero, angular momentum zero scattering length $a_0^0$. Since early experiments [4–8] observed only a few hundred events each, it was not until 1977, when the Geneva-Saclay experiment [9] gathered about 30,000 events, that a measurement was made of this quantity to 20% accuracy.

Since then no new data became available until Experiment 865 at the Brookhaven Alternate Gradient Synchrotron. From these data, the branching ratio (4.11 ± 0.11 ± 0.11) × 10$^{-5}$ and the $\pi\pi$ invariant mass dependence of the form factors $F$, $G$, and $H$ of the weak hadronic current as well as the phase shift difference $\delta^0_0 - \delta^1_1$ for $\pi\pi$ scattering were extracted. Using constraints based on analyticity and chiral symmetry, a new value with considerably improved accuracy for the s-wave $\pi\pi$ scattering length $a^0_0$ has been obtained also: $a^0_0 = 0.216 ± 0.013$ (stat) ± 0.002 (syst) ± 0.002 (theor).

We report experimental details and results of a new measurement of the decay $K^+\rightarrow\pi^+\pi^-e^+\nu_e$($K_{e4}$). A sample of more than 400,000 $K_{e4}$ events with low background has been collected by Experiment 865 at the Brookhaven Alternate Gradient Synchrotron. From these data, the branching ratio ($4.11 ± 0.01 ± 0.11)×10^{-5}$ and the $\pi\pi$ invariant mass dependence of the form factors $F$, $G$, and $H$ of the weak hadronic current as well as the phase shift difference $\delta^0_0 - \delta^1_1$ for $\pi\pi$ scattering were extracted. Using constraints based on analyticity and chiral symmetry, a new value with considerably improved accuracy for the s-wave $\pi\pi$ scattering length $a^0_0$ has been obtained also: $a^0_0 = 0.216 ± 0.013$ (stat) ± 0.002 (syst) ± 0.002 (theor).


FIG. 1. Kinematic quantities used in the analysis of $K_{e4}$ decay.

predictions for the scattering length. The tree level calculation $[\mathcal{O}(p^5)]$ [18]) yields $a^0_0 = 0.156$ (in this paper we use units of $m_N^2$ for the scattering length). The one-loop $[\mathcal{O}(p^4)]$, $a^0_0 = 0.201 \pm 0.01$ [19]) and the two-loop calculation $[\mathcal{O}(p^6)]$, $a^0_0 = 0.217$ [20]) show satisfactory convergence. The most recent calculation [21,22] matches the known chiral perturbation theory representation of the $\pi\pi$ scattering amplitude to two loops [20] with the dispersive representation that follows from the Roy equations [23,24], resulting in the prediction $a^0_0 = 0.220 \pm 0.005$. The high precision of this prediction has to be contrasted with the experimental value $a^0_0 = 0.26 \pm 0.05$ extracted from the Geneva-Saclay experiment [9] using the Roy equations and some peripheral $\pi N \to \pi \pi N$ data [25].

The form factors appearing in the weak hadronic current in the $K_{e4}$ decay matrix element have also been extensively used for the determination of the parameters of the ChPT Hamiltonian [26,27]. This program would clearly benefit from lower experimental uncertainties.

II. THEORETICAL BACKGROUND FOR THE ANALYSIS OF $K_{e4}$ DECAY

A. Kinematics

The decay

$$K^+(p) \rightarrow \pi^+(p_1) \pi^-(p_2) e^+(p_3) \nu_e(p_4)$$

(2)

can most conveniently be treated [28] by using three reference frames, as illustrated in Fig. 1: (1) the $K^+$ rest system ($\Sigma_k$), (2) the $\pi^+ \pi^-$ rest system ($\Sigma_{\pi\pi}$) and (3) the $e^+ \nu_e$ rest system ($\Sigma_{ee}$). The kinematics of the $K_{e4}$ decay are then fully described by five variables, introduced by Cabibbo and Maksymowicz [29]:

1. $s_\pi = M_{\pi\pi}^2$, the invariant mass squared of the dipion.
2. $s_e = M_{ee}^2$, the invariant mass squared of the dilepton.
3. $\theta_\pi$, the angle of the $\pi^+$ in $\Sigma_{\pi\pi}$ with respect to the direction of flight of the dipion in $\Sigma_k$.
4. $\theta_e$, the angle of the $e^+$ in $\Sigma_{ee}$ with respect to the direction of flight of the dilepton in $\Sigma_k$.
5. $\phi$, the angle between the plane formed by the two pions and the corresponding plane formed by the two leptons.

It is useful for the following discussion to introduce the combinations $P$, $Q$ and $L$ of the momentum four vectors $p_1$, $p_2$, $p_3$ and $p_4$ defined in Eq. (2) and two scalar products derived from them

$$P = p_1 + p_2, \quad Q = p_1 - p_2, \quad L = p_3 + p_4. \quad (3)$$

$$Q^2 = 4m_\pi^2 - s_\pi, \quad P \cdot L = \frac{1}{2}(m_K^2 - s_\pi - s_e). \quad (4)$$

$$X = [(P \cdot L)^2 - s_\pi s_e]^{1/2}, \quad \sigma = (1 - 4m_\pi^2/s_\pi)^{1/2}. \quad (5)$$

B. Matrix element

The matrix element is written as

$$M = \frac{G_F}{\sqrt{2}} V_{us}^* (p_\nu) \gamma_\mu (1 - \gamma_5) u(p_e)(V^\mu - A^\mu). \quad (6)$$

The vector current $V^\mu$ and the axial vector current $A^\mu$ have to be Lorentz invariant four-vectors:

$$A^\mu = \frac{1}{m_K} (FP^\mu + GQ^\mu + RL^\mu),$$

$$V^\mu = \frac{H}{m_K} e^{\mu\nu\rho\sigma} L_\nu P_\rho Q_{\sigma}. \quad (7)$$

The kaon mass $m_K$ was inserted to make the form factors $F$, $G$, $R$ and $H$ dimensionless complex functions of $p_1$, $p_2$, $p_1$, $p$ and $p_2$, $p$ or equivalently of $s_\pi$, $s_e$ and $\theta_\pi$.

C. Decay rate

The decay rate following from the matrix element given in Eq. (6) and neglecting terms proportional to $m_e^2/s_e$ is given by [30]

$$d\Gamma_5 = \frac{G_F^2 V_{us}^2}{128\pi^6 m_K^4} X \sigma J_5(s_\pi, s_e, \theta_\pi, \theta_e, \phi)$$

$$\times ds_\pi ds_e d(cos\theta_\pi) d(cos\theta_e) d\phi, \quad (8)$$

$$J_5 = I_1 + I_2 \cos\theta_e + I_3 \sin^2\theta_e \cos2\phi$$

$$+ I_4 \sin2\theta_e \cos\phi + I_5 \sin\theta_e \cos\phi + I_6 \cos\theta_e$$

$$+ I_7 \sin\theta_e \sin\phi + I_8 \sin2\theta_e \sin\phi$$

$$+ I_9 \sin^2\theta_e \sin2\phi. \quad (9)$$

Again neglecting terms proportional to $m_e^2/s_e$ the functions $I_1$ are given by

$$I_1 = \frac{1}{8} \{2|F_1|^2 + 3(|F_2|^2 + |F_3|^2)\sin^2\theta_\pi\}, \quad (10)$$

$$I_2 = -\frac{1}{8} \{2|F_1|^2 - (|F_2|^2 + |F_3|^2)\sin^2\theta_\pi\},$$

$$I_3 = -\frac{1}{4} \{(|F_2|^2 - |F_3|^2)\sin^2\theta_\pi, \quad I_4 = \frac{1}{2} \text{Re}(F_4^*F_2)\sin\theta_\pi,$$
\[ I_5 = -\text{Re}(F_1^* F_3) \sin \theta_\pi, \quad I_6 = -\text{Re}(F_1^* F_3) \sin^2 \theta_\pi, \]
\[ I_7 = -\text{Im}(F_1^* F_2) \sin \theta_\pi, \quad I_8 = -\frac{1}{2} \text{Im}(F_1^* F_3) \sin \theta_\pi, \]
\[ I_9 = -\frac{1}{2} \text{Im}(F_1^* F_3) \sin^2 \theta_\pi. \]

The form factors \( F, G, \) and \( H \) are contained in the functions \( F_i \), which are given by
\[
F_1 = X F + \sigma_q (P \cdot L) \cos \theta_\pi \cdot G,
\]
\[
F_2 = \sigma_q (s \cdot s_e) \sqrt{2} G,
\]
\[
F_3 = \sigma_q X (s \cdot s_e) \sqrt{2} \frac{H}{m_K^2}.
\]

The contribution of the form factor \( R \) is suppressed by a factor \( m_K^2/s \), and is therefore negligible. Consequently \( R \) cannot be determined from \( K_{e4} \) decay.

### D. Parametrization of the form factors

As noted above, the form factors \( F, G \) and \( H \) are functions of \( \theta_\pi, s_\pi \) and \( s_e \), and can be determined directly from a fit to the experimental data for sufficiently small bins of these kinematic variables. Alternatively a parametrization recently introduced by Amorós and Bijnen [31] may be used, which is based on a partial wave expansion in the variable \( \theta_\pi \):
\[
F = \left[ f_s + f_s' q^2 + f_s'' q^4 + f_s (s_\pi / 4m_\pi^2) \right] e^{i \delta_0(s_\pi)}
\]
\[
+ f_p' (\sigma_q X / 4m_\pi^2) \cos \theta_\pi e^{i \delta_1(s_\pi)},
\]
\[
G = \left[ g_p + g_p' q^2 + g_p' e (s_\pi / 4m_\pi^2) \right] e^{i \delta_0(s_\pi)},
\]
\[
H = \left( h_p + h_p' q^2 \right) e^{i \delta_1(s_\pi)},
\]
where \( q = [(s_\pi - 4m_\pi^2) / 4m_\pi^2]^{1/2} \) is the pion momentum in \( \Sigma_{s_\pi} \). This parametrization was constrained by theoretical models and the expected accuracy of the experimental data. It yields 10 new dimensionless form factor parameters \( f_s, f_s', f_s'', f_p', g_p, g_p', g_p, h_p, h_p' \), which do not depend on any kinematic variables, plus two phase shifts, which can be identified using Watson’s theorem [32] with the \( s \) and \( p \) wave (isoscalar and isovector, respectively) \( \pi \pi \) scattering wave shifts \( \delta_0 \) and \( \delta_1 \), which are still functions of \( s_\pi \).

In our analysis we will additionally assume \( f_s = f_s' = g_p = h_p' = 0 \). The validity of this assumption will be experimentally tested. When Eq. (12) is inserted into Eq. (11) and then into Eq. (10), it can be observed that the phase shift difference \( \delta = \delta_0 - \delta_1 \) enters via \( \cos \delta \) into the terms \( I_1, I_2, I_4, I_5 \) and via \( \sin \delta \) into the terms \( I_7 \) and \( I_8 \). Since \( \delta < 0.3 \) with \( \cos \delta > 0.95 \) holds in \( K_{e4} \) decay, and the kinematic factors suppress the term \( I_8 \), only the term \( I_7 \) is really relevant, which appears in the decay rate [Eq. (9) and Eq. (8)] multiplied by \( \sin \phi \). \( I_7 \) and \( I_8 \) are the only odd \( \phi \) terms. Hence, as noted by Shabalin [1], and Pais and Treiman [30], the asymmetry of the \( \phi \) distribution is the observable that is most sensitive to the phase shifts. This also holds for any other parametrization of the form factors. The amplitude of the asymmetry is quite small compared to the \( \phi \) independent part, as Figs. 6 and 9 illustrate. This explains why a very high statistics data sample is needed for an accurate measurement of the phase shift difference.

### E. \( \pi \pi \) scattering length

To establish a relation between the phase shift \( \delta_0^0 \) and the scattering length normally the analytical properties of the \( \pi \pi \) scattering amplitudes and crossing relations are used, which lead to dispersion relations contained in the Roy equations [23]. Ananthanarayan et al. [24] have recently updated earlier treatments [33], which were used in the analysis of \( \pi \pi \) scattering data, and solved these equations numerically. Their analysis made use of a phase shift parametrization originally proposed by Schenk [34]:
\[
\tan \delta_1^0 = \sqrt{1 - \frac{4m_\pi^2}{s_\pi} q^2 f(A_1^i + B_1^i q^2 + C_i q^4 + D_i q^6)} \times \left( \frac{4m_\pi^2 - s_\pi}{s_\pi - s_\ell} \right),
\]

The solution of the Roy equations implies that the parameters \( s_\ell^i, A_1^i, B_1^i, \) etc. can be expressed as a function of only two parameters or subtraction constants, which are identified as the \( I = 0 \) and \( I = 2 \) waving scattering lengths \( a_0^0 \) and \( a_0^2 \). For example, the first two coefficients of this expression for the \( I = \ell = 0 \) case read as follows [35]:
\[
A_0^0 = a_0^0,
\]
\[
B_0^0 = 0.2395 + 0.9237 \Delta a_0^0 - 3.352 \Delta a_0^2 + 0.2817 (\Delta a_0^0)^2 + 6.335 (\Delta a_0^0)^2 + 6.074 (\Delta a_0^2)^2 \ldots.
\]
\[
s_0^0 = 36.83 m_\pi^2 (1 + 0.2764 \Delta a_0^0 - 0.1409 \Delta a_0^2 + \ldots)
\]
\[
= (0.847)^2 \text{ GeV}^2,
\]

where \( \Delta a_0^0 = a_0^0 - 0.220 \) and \( \Delta a_0^2 = a_0^2 + 0.444 \). Although \( K_{e4} \) decay allows only \( I = 0 \) and \( I = 1 \) contributions, the use of the crossing relations brings in a modest dependence on the \( I = 2 \) scattering length. The \( I = 1 \) phase shifts at low energies are dominated by the \( \rho \) resonance and are furthermore small in the region of interest for \( K_{e4} \).

It was recognized by Morgan and Shaw [36] that the possible values of \( a_0^0 \) and \( a_0^2 \) are restricted to a band in the \( a_0^0 - a_0^2 \) plane, the so-called universal band. This band is defined as the area which is allowed by \( \pi \pi \) scattering data above 0.8 GeV [37,38] and the Roy equations. The allowed range, estimated in the most recent analysis [24], is shown in Fig. 10. The central curve of this band is given by
where the figure given in the bracket indicates the width of the band. Figure 2 illustrates the influence of the universal band and how the phase shift difference \( \delta = \delta_0 - \delta_1 \) depends on the scattering length \( a_0 \).

It has recently been shown by Colangelo et al. [22,39] that the width of the allowed band can be considerably reduced to \( [\pm 0.0008] \), if chiral symmetry constraints are imposed in addition. \( a_0 \) and \( a_0^3 \) are then related as

\[
\Delta a_0^2 = 0.236 \Delta a_0^3 - 0.61 (\Delta a_0^3)^2 - 9.9 (\Delta a_0^3)^3,
\]

where \( \Delta a_0^2 \) and \( \Delta a_0^3 \) have been defined above. This band is also depicted in Fig. 10 with the label CLG.

In ChPT up to order \( O(p^3) \) the scattering lengths are linked to two coupling constants \( \ell_3 \) and \( \ell_4 \). For example, \( \ell_3 \) determines the size of the first order correction to the Gell-Mann–Oakes–Renner relation [Eq. (1)] [15], and is assumed to be \textit{a priori} unknown in GChPT. Colangelo et al. [22,39] have argued, that both \( a_0^3 \) and \( a_0^5 \) can be made dependent solely on \( \ell_3 \), if the scalar radius of the pion is used as an additional input to give a relation between \( \ell_3 \) and \( \ell_4 \). This also holds in GChPT, and Eq. (15) results, when \( \ell_4 \) is eliminated. Once the scattering lengths are known experimentally, a constraint for \( \ell_3 \) and consequently for the quark condensate can be derived.

III. EXPERIMENTAL SET UP

A. Apparatus

The analysis outlined here is based on data recorded at the Brookhaven Alternate Gradient Synchrotron (AGS) in a dedicated run at reduced beam intensity in 1997, employing the E865 detector. The apparatus, described in great detail in [40], is shown in Fig. 3. Here we will mention only its main features. The detector was located in a 6 GeV unseparated beam of approximately \( 1.5 \times 10^7 \) \( K^+ \) accompanied by about \( 3 \times 10^8 \) \( \pi^+ \) and protons per machine spill of \( 1.6-2.8 \) s duration. About 6% of the kaons accepted by the beam line decayed in the 5 m long evacuated decay volume. The decay products were separated by charge and swept away from the beam by a first dipole magnet. Negatively charged particles were deflected to the left. A second dipole magnet sandwiched between four proportional wire chambers (P1-P4) served as the spectrometer. The wire chambers, each consisting of four wire planes, were deadened in the region where the beam passed. This arrangement yielded a momentum resolution of \( \sigma_p = 0.003 \) \( P_2 \) GeV/c, where \( P \), the momentum of the decay products in GeV/c, had a typical range of 0.6 to 3.5. Pions and muons were distinguished from positrons and electrons using two Čerenkov counters, C1 and C2, situated inside and behind the second dipole magnet, and rendered insensitive in the beam region. Both Čerenkov counters, when filled with CH, at atmospheric pressure, yielded on average seven photoelectrons, and hence ensured an electron identification probability greater than 99%. An electromagnetic calorimeter of the Shashlyk design [41], located downstream of P4 further aided the separation of the positrons from other charged decay products. It consisted of 30 modules in the horizontal and 20 modules in the vertical direction, but for the beam region, where \( 6 \times 3 \) modules were absent. Module size was 11.4 cm high and 11.4 cm wide perpendicular to the beam direction and 15 radiation length deep. The calorimeter was followed by an array of 12 muon chambers, separated by iron planes, employed to discriminate pions against muons. Four hodoscopes were added to the detector for trigger purposes. The A hodoscope was situated just upstream of the calorimeter, the B- and C-hodoscopes were embedded in the muon stack, and the D-hodoscope was located between the first two proportional wire chambers. The detector was completed by a pixel

\[ \delta_0 - \delta_1 \]

\[ M_{\pi\pi} \text{ [MeV]} \]
TABLE I. Experimental resolutions for the five kinematic variables used in the analysis.

<table>
<thead>
<tr>
<th>Variable</th>
<th>FWHM</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s$</td>
<td>0.00133 GeV$^2$</td>
</tr>
<tr>
<td>$s'$</td>
<td>0.00361 GeV$^2$</td>
</tr>
<tr>
<td>$\theta_s$</td>
<td>0.147 rad</td>
</tr>
<tr>
<td>$\theta_e$</td>
<td>0.111 rad</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.404 rad</td>
</tr>
</tbody>
</table>

counter, installed just upstream of the decay volume, which measured the position of the incoming kaons. This device consisted of an array of 12 (horizontally) by 32 (vertically) scintillating pixels, each with an area of $7 \times 7$ mm$^2$.

Table I summarizes the resolution of the apparatus in the five variables required to describe the kinematics of the $K_{e4}$ decay.

B. Trigger requirements

The trigger was designed as a multilevel structure with increasing sophistication. The lowest trigger level (T0) indicated the presence of three charged particle tracks, two on the right and one on the left side, each signaled by a coincidence between the A counter and the corresponding calorimeter module directly behind it (A·SH). For each combination of coincidences on the right only a limited, kinematically acceptable region on the left was allowed. To ensure that the trigger resulted from particles coming from the decay volume, at least one coincidence on both sides between the D-counter and A·SH was required. The next trigger level (T1) demanded the presence of a positron in order to reject events from the $K^+ \rightarrow \pi^+ \pi^+ \pi^- (K_e)$ decay, and dismissed all events with evidence for the presence of an electron to eliminate events from $K^+ \rightarrow \pi^+ \pi^0 (\pi^0 \rightarrow e^+ e^- \gamma, K_{dal})$ decay, both rather common decay modes. Consequently, this trigger level required signals in both Čerenkov counters on the right (corresponding to at least 2.5 photoelectrons) and vetoed all events with a signal in either Čerenkov counters on the left (at least 0.25 photoelectrons). The final trigger level (T2) rejected events with a high occupancy in the wire chambers, most likely caused by noise in the read-out electronics. It did not reject many events, but the ones it rejected would have required an exceedingly large amount of computer time in the reconstruction. In addition to $K_{e4}$ candidates, a few prescaler monitor triggers were also recorded, e.g. a minimum bias trigger (T0 without the T1 requirement) dominated by accidentals and $K_e$ events, and a trigger sensitive to $K_{dal}$ events, used to check the Čerenkov counter efficiency [40].

IV. $K_{e4}$ EVENT SELECTION AND ANALYSIS

A. Reconstruction

The kinematic reconstruction of an event, described in detail in [40], proceeded as follows: In the first step raw wire hits in the proportional chambers were combined to space points, requiring signals in at least three of the four wire planes in a chamber. Then the space points were combined to tracks. A track was found if at least three chambers contributed with a space point each. Next, employing a measured map of the magnetic field in the dipole magnets, the momenta of the tracks were fitted. For events with at least three reconstructed tracks, a fitting algorithm, again utilizing the field map, determined the decay vertex as the position from which the distance $s$ to the three tracks was minimal. For events containing more than three tracks, the combination that produced the lowest $s$ was tagged as the most probable set of track candidates from kaon decay. Finally, the kaon direction was obtained from the hit in the pixel counter and the vertex. The kaon momentum could then be fitted by tracing the kaon back through the beam line to the production target 27.5 m upstream of the decay tank. In the last reconstruction step the particle identification information was assigned to the tracks found.

B. Selection

$K_{e4}$ candidates had to pass the following selection criteria: a vertex within the decay tank of acceptable quality $s$, a momentum reconstructed from the three daughter particles below the beam momentum, a timing spread between the signals caused by the tracks in the A-hodoscope and the calorimeter consistent with the resolution of 0.5 ns. Finally we required an unambiguous identification of the $e^+$, assured by light in the appropriate photomultiplier tubes in both Čerenkov counters and an energy loss in the calorimeter consistent with the momentum of the track, and of the $\pi^-$, secured by the absence of a signal above the noise in the Čerenkov counters and an energy loss in the calorimeter consistent with that of a minimum ionizing particle or a hadron shower. The cuts described above ensured $K_{e4}$ events of good quality, but the resulting event sample still contained a considerable amount of background events.

C. Backgrounds

The major background contributions came from $K_e$ decay and accidentals. A $K_e$ could fake a $K_{e4}$ by either (1) a misidentification of one of the $\pi^0$ as a positron due to $\delta$ rays, noise in the photomultiplier tubes or the presence of an additional parasitic positron, or (2) a decay of a $\pi^-$ directly or via a $\mu^-$ into an $e^+$. The dominating accidental background arose from combinations of a $\pi^+$ and a $\pi^-$ originating from a $K_e$ decay with a positron from either the beam or from a $K_{dal}$ decay (2-1 accidental from $K_e$).

To reject background from $K_e$ decay, we required that the kaon reconstructed from the three charged daughter particles did not track back to the target, using the fact that the reconstruction for $K_{e4}$ is incomplete due to the undetected neutrino. The remaining $K_e$ background can be made visible by plotting the $K_{e4}$ candidates under the $K_e$ hypothesis, i.e. assigning to the positron a pion mass. The $K_e$ background appears as a narrow peak sitting on the broad distribution originating from $K_{e4}$ decays, as seen in Fig. 4(a).

Accidentals of the 2-1 type from $K_e$ are characterized by (1) the positron track tends to be out of time in the A-hodoscope and the calorimeter compared with the two
pion tracks; (2) the distance of closest approach between the positron track and each pion track is typically larger than the distance between the two pions; (3) the position of the vertex along the beam axis tends to be more upstream in $K_\tau$ and hence also in 2-1 accidentals from $K_\tau$ compared with $K_{e4}$, due to smaller average transverse momentum; (4) in the calorimeter more clusters of energy are found, due to the possibility of two decays in the same time window. These characteristics were used to construct a likelihood function in order to suppress 2-1 accidentals. The remaining background can be exposed by inspecting the distribution of the total visible momentum in the event, reconstructed from the sum of the three charged particle momenta. Accidentals of the 2-1 type display a large tail above the beam momentum, as is demonstrated in Fig. 4(b). The agreement between data and the sum of Monte Carlo and background indicates that this background is well understood. For the background simulation we used $K_\tau$ monitor events with a fourth accidental positron track. The uncertainty in the evaluation of this background under the signal region below the beam momentum yields the largest contribution to the systematic error of the background estimate.

The excellent particle identification capabilities of our apparatus reduce the background originating from $K_{dal}$ decay, where the $e^-$ gets misidentified as a $\pi^-$, to a negligible level. This can be made evident by plotting the invariant mass $M_{ee}$ of the electron-positron pair, assigning the electron mass to the reconstructed $\pi^-$ for $K_{e4}$ events. $K_{dal}$ events (inset) are characterized by low values of $M_{ee}$.

V. MONTE CARLO SIMULATION

A good Monte Carlo simulation of the detector is a necessary ingredient for the analysis of the decay distributions and the determination of the absolute branching ratio. This simulation starts with the kaon beam at the upstream end of the decay tank with a spatial and momentum distribution deduced from our ample supply of $K_\tau$ monitor events, for which the incident $K^+$ can be fully reconstructed. The $K^+$ is then allowed to decay in a preselected mode along its trajectory in the decay tank. To model the physics of the $K_{e4}$ decay, initial values of the matrix elements were chosen in accordance with the ChPT analysis at the one loop level [42,43] of the Geneva-Saclay experiment [9]. Radiative corrections are included following Diamant-Berger [44] (see also Sec. VII below). For the decay modes $K_\tau$ and $K_{dal}$,

Table II summarizes the background rates.

D. Final sample

After applying the event selection criteria described above, 406,103 events remained, of which we estimate 388,270 $\pm$ 5025 to be $K_{e4}$ events. This corresponds to an increase in statistics by more than a factor of 10 compared with previous experiments.

TABLE II. Compilation of fraction of background events. 1-1-1 accidentals: accidental combinations of two independent pion tracks and a positron track; 2-1 accidentals: combinations of two pions from a $K_\tau$ with an accidental positron or combinations of a $\pi^+$ and a positron from a $K_{dal}$ decay with an accidental $\pi^-$; $[a]$ $\pi^0 \rightarrow e^+ e^- \gamma$ and $e^-$ misidentification.

<table>
<thead>
<tr>
<th>Background</th>
<th>Fraction</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_\tau$ with $\pi^+$ misidentification</td>
<td>$(1.3 \pm 0.3) \times 10^{-2}$</td>
</tr>
<tr>
<td>$K_\tau$ with $\pi^+ \rightarrow e^+ \nu_e$</td>
<td>$(3.5 \pm 0.2) \times 10^{-3}$</td>
</tr>
<tr>
<td>$K_\tau$ with $\pi^+ \rightarrow \mu^+ \nu_\mu$ and $\mu^+ \rightarrow e^+ \nu_\mu \bar{\nu}_\mu$</td>
<td>$(2.6 \pm 0.3) \times 10^{-3}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 \pi^+ [a]$</td>
<td>$(2.5 \pm 0.6) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 e^+ \nu_e [a]$</td>
<td>$(0.4 \pm 0.1) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^0 \mu^+ \nu_\mu [a]$</td>
<td>$(0.4 \pm 0.1) \times 10^{-5}$</td>
</tr>
<tr>
<td>$K^+ \rightarrow \pi^+ \pi^- \pi^0 [a]$</td>
<td>$(0.3 \pm 0.1) \times 10^{-5}$</td>
</tr>
<tr>
<td>$1-1-1$ accidentals</td>
<td>$(0.9 \pm 0.4) \times 10^{-4}$</td>
</tr>
<tr>
<td>2-1 accidentals from $K_\tau$</td>
<td>$(2.4 \pm 1.2) \times 10^{-2}$</td>
</tr>
<tr>
<td>2-1 accidentals from $K_{dal}$</td>
<td>$(0.9 \pm 0.4) \times 10^{-3}$</td>
</tr>
</tbody>
</table>
needed for the determination of the branching ratio and the evaluation of the background, we use the matrix elements given in Refs. [45] and [46]. The detector response is handled with a GEANT-based [47] simulation of the E865 apparatus, and the simulated events are processed through the same reconstruction and selection programs as data events. With these tools, we generated $8.16 \times 10^6$ $K_{e4}$ events, resulting in $2.9 \times 10^6$ accepted events, about 7.5 times more than data events. The quality of the simulation is demonstrated in Fig. 5, which displays the vertex quality $s$, the missing neutrino mass squared, and the position of the vertex along the beam axis as examples. The vertex quality is a crucial quantity in the event reconstruction; the missing neutrino mass squared is sensitive to the resolution; and the vertex position depends on the decay matrix element and detector acceptance. The good agreement between data and Monte Carlo indicates that ChPT describes the data well and that our event selection procedure did not introduce a significant bias. We also compare Monte Carlo with data distributions for the kinematically very distinct $K_\tau$ and $K_{dal}$ decays, getting again a nice agreement (see, e.g. [40]). Furthermore, we find that the $K_{dal}$ branching ratio is consistent with the published value [48], using $K_\tau$ as normalization channel. This underlines the good understanding of the geometrical acceptance and the efficiency of the various detector elements.

**VI. BRANCHING RATIO**

The $K_{e4}$ branching ratio was normalized with respect to the $K_\tau$ decay. As mentioned in Sec. III B, we collected $K_\tau$ events in a minimum bias trigger concurrently with $K_{e4}$ events. $K_\tau$ is the most common kaon decay with three charged particles in the final state, which strongly simplifies the selection of a clean sample of events. To identify $K_\tau$ events, we require the reconstruction of a vertex, as for $K_{e4}$, and the reconstruction of the kaon mass. With $BR(\tau) = 5.59 \pm 0.05%$ [48], the $K_{e4}$ branching ratio $BR$ and the decay rate $\lambda$ are calculated as

$$BR(K_{e4}) = BR(K_\tau) \frac{N(K_{e4})A(K_\tau)}{N(K_\tau)A(K_{e4})} C$$

$$N(K_{e4})[N(K_\tau)] = \text{number of } K_{e4}[K_\tau] \text{ events}$$

$$= 388,270 \pm 5025 [1.487 \times 10^9]$$

$$A(K_{e4})[A(K_\tau)] = \text{acceptance for } K_{e4}[K_\tau] \text{ events}$$

$$= 3.77\% \ [10.29\%]$$

$$C = \text{accidental veto correction} = 1.0312 \pm 0.0022,$$

leading to

$$BR(K_{e4}) = (4.109 \pm 0.008 \pm 0.110) \times 10^{-5}$$

$$\lambda(K_{e4}) = (3321 \pm 6 \pm 89) \text{s}^{-1}.$$
TABLE III. $K_{e4}$ branching ratios measured in older experiments.

<table>
<thead>
<tr>
<th>Reference</th>
<th>No. of events</th>
<th>Branching ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>PDG [48]</td>
<td></td>
<td>$(3.91 \pm 0.17) \times 10^{-5}$</td>
</tr>
<tr>
<td>Rosselet et al. [9]</td>
<td>30318</td>
<td>$(4.03 \pm 0.17) \times 10^{-5}$</td>
</tr>
<tr>
<td>Beier et al. [8]</td>
<td>8141</td>
<td>$(3.91 \pm 0.50) \times 10^{-5}$</td>
</tr>
<tr>
<td>Bourquin et al. [7]</td>
<td>1609</td>
<td>$(4.11 \pm 0.38) \times 10^{-5}$</td>
</tr>
<tr>
<td>Schweinberger et al. [6]</td>
<td></td>
<td>$(3.26 \pm 0.35) \times 10^{-5}$</td>
</tr>
<tr>
<td>Ely et al. [5]</td>
<td>269</td>
<td>$(3.74 \pm 0.84) \times 10^{-5}$</td>
</tr>
<tr>
<td>Birge et al. [4]</td>
<td>69</td>
<td>$(3.47 \pm 0.84) \times 10^{-5}$</td>
</tr>
</tbody>
</table>

VII. FITS TO THE DECAY DISTRIBUTIONS

In pursuing the goal of determining the form factors and $\pi\pi$ scattering phase shifts, three different approaches have been followed, which have been outlined in Sec. II D. The $K_{e4}$ form factors $F$, $G$, and $H$, and the phase shift $\delta$ can be directly extracted for a conveniently chosen grid of bins in the kinematic variables. This approach makes no assumption on the analytical behavior of these quantities. In the second approach, the parametrization of Eq. (12) is used and the phase shifts are related to the two scattering lengths using Eq. (13). This allows use of the whole data sample in a single fit. Finally, either Eq. (14) or Eq. (15) can be used in addition, reducing the number of parameters by one. The statistical method which we describe below is the same for all three approaches.

TABLE IV. Systematic errors in the branching ratio measurement.

<table>
<thead>
<tr>
<th>Sources</th>
<th>$\sigma_{BR}/BR$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Background subtraction</td>
<td>0.012</td>
</tr>
<tr>
<td>$K_{e4}$ prescale factor</td>
<td>0.0076</td>
</tr>
<tr>
<td>Magnetic field map</td>
<td>0.005</td>
</tr>
<tr>
<td>Čerenkov counterinefficiencies</td>
<td>0.015</td>
</tr>
<tr>
<td>PWC efficiencies</td>
<td>0.006</td>
</tr>
<tr>
<td>Fiducial volume</td>
<td>0.005</td>
</tr>
<tr>
<td>Track quality</td>
<td>0.0022</td>
</tr>
<tr>
<td>Vertex reconstruction</td>
<td>0.0016</td>
</tr>
<tr>
<td>Z position of vertex</td>
<td>0.0012</td>
</tr>
<tr>
<td>Tracking back to target</td>
<td>0.0019</td>
</tr>
<tr>
<td>Timing cuts</td>
<td>0.0020</td>
</tr>
<tr>
<td>$e^-$ identification in the calorimeter</td>
<td>0.0007</td>
</tr>
<tr>
<td>$\pi^-$ identification</td>
<td>0.0011</td>
</tr>
<tr>
<td>2-1 accidental likelihood</td>
<td>0.0006</td>
</tr>
<tr>
<td>$K_{e4}$ matrix element (statistics)</td>
<td>0.006</td>
</tr>
<tr>
<td>$K_{e4}$ mass resolution</td>
<td>0.0081</td>
</tr>
<tr>
<td>$K_{e4}$ branching ratio</td>
<td>0.009</td>
</tr>
<tr>
<td>Total (added quadratically)</td>
<td>0.0268</td>
</tr>
</tbody>
</table>

A. Data treatment

The experimental distributions must be fit to Eq. (8), taking into account the acceptance and resolution of the apparatus, with the form factors and phase shifts as free parameters. Following the recommendations by Eadie [49] we select equi-probable bins for each kinematic variable, namely six bins in $s_\gamma$, five in $s_e$, ten in $\cos \theta_\pi$, six in $\cos \theta_e$, and 16 bins in $\phi$. With a total of 28,800 bins there are on average 13 events in each bin.

Following the procedure used by the Geneva-Saclay experiment [9,44], we minimize a $\chi^2$ function defined as

$$
\chi^2 = 2 \sum_j n_j \ln \left( \frac{n_j}{r_j} \left( 1 - \frac{1}{m_j + 1} \right) \right) + 2 \sum j (n_j + m_j + 1) \ln \left( \frac{n_j}{1 + \frac{r_j}{m_j + 1}} \right),
$$

where the sum runs over all bins. $n_j$, $r_j$, and $m_j$ are the number of data events, expected events and generated Monte Carlo events in bin $j$, respectively. This $\chi^2$ is deduced from the probability

$$
P(n,m,r) = \int_0^{\infty} \int_0^{\infty} e^{-n/m} e^{-v/m} v^m \delta \left( u - \frac{r}{v} \right) dudv
$$

and takes into account the limited number of Monte Carlo events. It reduces to the more familiar expression

$$
\chi^2 = \sum_j \left( 2(r_j - n_j) + 2n_j \ln(n_j/r_j) \right)
$$

for large $m_j$.

The expected number of events in $r_j$ is calculated to be

$$
r_j = Br(K_{e4}) \frac{N^K}{N^{MC}} \sum J_j^K(F,G,H)^{new} / J_j^K(F,G,H)^{MC},
$$

where the sum runs over all Monte Carlo events in bin $j$. $N^K$ is the number of $K^+$ decays derived from the number of $K_\tau$ events. $N^{MC}$ is the number of generated events. $J_j^K(F,G,H)^{MC}$ [Eq. (9)] is evaluated at the relevant set of kinematic variables for the simulated event with the form factors $F$, $G$, and $H$ calculated at $q = q^{MC}$. $J_j^K(F,G,H)^{new}$ is evaluated with the same kinematic set and $F$, $G$, $H$ recalculated from the parameters of the fit. Thus, we apply the parameters on an event by event basis, and at the same time, we divide out a possible bias caused by the matrix element, making the fit independent of the ChPT ansatz used to generate the Monte Carlo events.

B. Fit of the decay rate in multiple bins in $s_\gamma$

For the fit in multiple bins two further assumptions are being made, namely that the form factors do not depend on $s_e$ and that the form factor $F$ contributes to $s$ waves only.
This is equivalent to setting $f_\epsilon$, $g_\epsilon$ and $\tilde{f}_\rho$ equal to zero in the parametrization of Ref. [31]. The validity of these assumptions will be discussed in Sec. VII C below. Hence the contributions to the systematic error of that of the branching ratio measurement. The major contributions to the systematic error come from the background, and resolution effects, i.e. deviations between the recommendations by Lafferty and Wyatt [53] which illustrates the high quality of the fit.

The centroids $\langle M_{\pi\pi} \rangle$ of the bins are estimated following the recommendations by Lafferty and Wyatt [50]. The dominant systematic error for $F$, $G$, and $H$ has the same origin as that of the branching ratio measurement. The major contributions to the systematic error of $\delta$ are the subtraction of the background, and resolution effects, i.e. deviations between the original and reconstructed kinematics.

We have also included the full magnitude of the radiative corrections in the systematic error. As mentioned above in Sec. V, we have calculated these corrections using formulas given in Refs. [44,51] based on the work of Neveu and Scherk [52]. Basically one has to consider two types of radiative corrections, those where a real photon is radiated by one of the charged particles involved in the decay and those where a virtual photon is exchanged between two charged particles. The former are dominated by inner bremsstrahlung in particular of the positron [44], as e.g. experimentally determined in the related decay $K_{L3}^{+} \rightarrow \pi^{+}e^{+}\nu_{e}(\nu_{\mu})$ [53]. The Low theorem [54] ensures that off-shell effects appear only in second order and hence modifications of the hadronic form factors are expected to be negligible. The Coulomb interaction of the charged particles in the decay, however, has noticeable effects, in particular its most important contribution, the mutual attraction of the pion pair, has already been observed in the Geneva-Saclay experiment [9,44]. The repulsion or attraction between the positron, kaon and the two pions, which we also included, is unimportant. As an example we have reproduced the $\pi\pi$ Coulomb attraction below [51], which we have used to reweight each event:

$$d\Gamma_T = d\Gamma_0 (1 + \alpha C),$$

where

$$C = \pi \frac{1 + v^2}{2v} + 2 \ln \left( \frac{2E_\pi}{m_\pi} \right) \left[ \frac{1 + v^2}{2v} \ln \left( \frac{1 + v}{1 - v} \right) - 1 \right]$$

$$+ \frac{1}{\pi} \left( \frac{2 + v^2}{2v} \right) \ln \left( \frac{1 + v}{1 - v} \right) + 8A \frac{1 + v^2}{2v} - 1,$$

and

$$A = \int_0^{0.5 \ln((1 + v)/(1 - v))} z \coth z dz$$

$$= \mathcal{L}_2(v) - \mathcal{L}_2(-v) - \frac{1}{2} \left[ \mathcal{L}_2 \left( \frac{2}{1 + v} \right) - \mathcal{L}_2 \left( \frac{2}{1 - v} \right) \right].$$

<table>
<thead>
<tr>
<th>$M_{\pi\pi}$, $\langle M_{\pi\pi} \rangle$ (MeV)</th>
<th>280–294, 285.2</th>
<th>294–305, 299.5</th>
<th>305–317, 311.2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>5832±13±80 (−26)</td>
<td>5875±14±83 (−34)</td>
<td>5963±14±90 (−44)</td>
</tr>
<tr>
<td>$G$</td>
<td>4703±89±69 (−22)</td>
<td>4694±62±67 (−27)</td>
<td>4772±54±70 (−34)</td>
</tr>
<tr>
<td>$H$</td>
<td>−3740±800±180 (−59)</td>
<td>−3500±520±190 (−50)</td>
<td>−3550±440±200 (−167)</td>
</tr>
<tr>
<td>$\delta = \delta^0_0 - \delta^1_0$</td>
<td>−16±40±2 (+0.5)</td>
<td>68±25±1 (−0.4)</td>
<td>134±19±2 (−1.3)</td>
</tr>
<tr>
<td>$\chi^2$/NDF</td>
<td>1.071</td>
<td>1.080</td>
<td>1.066</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$M_{\pi\pi}$, $\langle M_{\pi\pi} \rangle$ (MeV)</th>
<th>317–331, 324.0</th>
<th>331–350, 340.4</th>
<th>&gt;350, 381.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$F$</td>
<td>6022±16±94 (+46)</td>
<td>6145±17±96 (+45)</td>
<td>6196±20±83 (+34)</td>
</tr>
<tr>
<td>$G$</td>
<td>5000±51±82 (+38)</td>
<td>5003±49±83 (+31)</td>
<td>5105±50±74 (+31)</td>
</tr>
<tr>
<td>$H$</td>
<td>−3630±410±230 (−177)</td>
<td>−1700±410±240 (−160)</td>
<td>−2230±480±330 (−173)</td>
</tr>
<tr>
<td>$\delta = \delta^0_0 - \delta^1_0$</td>
<td>160±17±2 (+0.1)</td>
<td>212±15±3 (+0.2)</td>
<td>284±14±3 (+0.6)</td>
</tr>
<tr>
<td>$\chi^2$/NDF</td>
<td>1.103</td>
<td>1.093</td>
<td>1.034</td>
</tr>
</tbody>
</table>

FIG. 6. $\phi$ distributions for the six bins in $M_{\pi\pi}$. The markers with error bars represent the data, the histogram the modified Monte Carlo distribution after the fit.
TABLE VI. Form factors and scattering length $a_0^g$ in the parametrization of Eq. (12) using either Eq. (14) or Eq. (15). The results for the form factor parameters are identical for both fits. The first error is statistical, the second systematic. The quantity in parentheses is the shift in the result of the parameter which resulted from the radiative corrections.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>$\chi^2$/NDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_s$</td>
<td>$5.75 \pm 0.02 \pm 0.08$ ($-0.03$)</td>
<td></td>
</tr>
<tr>
<td>$f'_s$</td>
<td>$1.06 \pm 0.10 \pm 0.40$ ($+0.37$)</td>
<td></td>
</tr>
<tr>
<td>$f''_s$</td>
<td>$-0.59 \pm 0.12 \pm 0.40$ ($-0.37$)</td>
<td></td>
</tr>
<tr>
<td>$g_p$</td>
<td>$4.66 \pm 0.05 \pm 0.07$ ($+0.03$)</td>
<td></td>
</tr>
<tr>
<td>$g''_p$</td>
<td>$0.67 \pm 0.10 \pm 0.04$ ($+0.00$)</td>
<td></td>
</tr>
<tr>
<td>$h_p$</td>
<td>$-2.95 \pm 0.19 \pm 0.20$ ($-0.16$)</td>
<td></td>
</tr>
<tr>
<td>$a^0_0$</td>
<td>$0.228 \pm 0.012 \pm 0.004$ ($\pm 0.000$) [Eq. (14)]</td>
<td></td>
</tr>
<tr>
<td>$a^0_0$</td>
<td>$0.216 \pm 0.013 \pm 0.004$ ($\pm 0.000$) [Eq. (15)]</td>
<td></td>
</tr>
</tbody>
</table>

The small deviation of $\chi^2$/NDF from the expected value of one may reflect the discreteness of the background. The number of background events which we add to the generated events is smaller than the number of bins, and the background is distributed over almost the whole phase space. By using tighter cuts, which reduce the background contributions by a factor of two, we have confirmed that the results for the form factors and phase shifts remain unchanged.

The results from Table V allow us to examine the $s_\pi$ dependence of the form factors $F$ and $G$, and of the phase shifts $\delta$, which are displayed in Figs. 7 and 8. For the various fits to these data, which we report below, the value of $\chi^2$/NDF is always below one. Following Amorós and Bijnen [31], we fitted $F$ with a second degree polynomial, while a linear function suffices for $G$, with the following results:

$$ f_s = 5.77 \pm 0.10, \quad f'_s = 0.95 \pm 0.58, \quad f''_s = -0.52 \pm 0.61, $$

$$ g_p = 4.68 \pm 0.09, \quad g'_p = 0.54 \pm 0.20. $$

TABLE VII. Results from the fits, where the form factors parameters $f_p$, $f_e$, and $g_e$ were allowed to vary one at a time. The quantity in parentheses shows the influence of radiative corrections.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>$\chi^2$/NDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_p$</td>
<td>$-0.34 \pm 0.10 \pm 0.27$ ($-0.02$)</td>
<td>30952/28792</td>
</tr>
<tr>
<td>$f_e$</td>
<td>$-0.32 \pm 0.10 \pm 0.24$ ($+0.02$)</td>
<td>30954/28792</td>
</tr>
<tr>
<td>$g_e$</td>
<td>$0.04 \pm 0.34 \pm 0.88$ ($\pm 0.00$)</td>
<td>30963/28792</td>
</tr>
</tbody>
</table>

$\mathcal{L}^2(x) = -\int_0^y \frac{1}{y} \ln(1-y)dy.$

Here $v$ is the velocity of the pions in the dipion center-of-mass system (in units of $c$), $\alpha$ the fine-structure constant, and $E_{\text{cm}}$ a cut-off energy fixed at 30 MeV. In all tables where results are given (Tables V, VI, VII and VIII) we have listed the effect of applying the radiative corrections separately. While the form factors $F$ and $G$ and the phase shifts $\delta$ are nearly unaffected, the form factor $H$ changes between 1.5 and 9.4%.

The quantity in parentheses shows the influence of the radiative corrections. The results in Table VIII are given in units of $H^2$. The first error is statistical, the second systematic. The quantity in parentheses shows the influence of the radiative corrections.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
<th>$\chi^2$/NDF</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f_s$</td>
<td>$5.75 \pm 0.02 \pm 0.08$ ($-0.03$)</td>
<td>30963/28792</td>
</tr>
<tr>
<td>$f'_s$</td>
<td>$1.06 \pm 0.10 \pm 0.40$ ($+0.37$)</td>
<td>30963/28792</td>
</tr>
<tr>
<td>$f''_s$</td>
<td>$-0.60 \pm 0.12 \pm 0.40$ ($-0.37$)</td>
<td>30963/28792</td>
</tr>
<tr>
<td>$g_p$</td>
<td>$4.65 \pm 0.48 \pm 0.07$ ($+0.03$)</td>
<td>30963/28792</td>
</tr>
<tr>
<td>$g'_p$</td>
<td>$0.09 \pm 0.11 \pm 0.04$ ($+0.00$)</td>
<td>30963/28792</td>
</tr>
<tr>
<td>$h_p$</td>
<td>$-2.95 \pm 0.19 \pm 0.20$ ($-0.16$)</td>
<td>30963/28792</td>
</tr>
<tr>
<td>$a^0_0$</td>
<td>$0.203 \pm 0.033 \pm 0.004$ ($-0.001$)</td>
<td>30963/28792</td>
</tr>
</tbody>
</table>

Figure 7 also shows the results of a linear fit: $F(q) = F(0) \times (1 + \lambda_F q^2)$. We found

$$ F(0) = 5.83 \pm 0.08, \quad \lambda_F = 0.079 \pm 0.015, $$

where the error of $\lambda_F$ was calculated using only the relative errors of $F$ in the six bins. These results are in agreement with those of the Geneva-Saclay experiment [9], namely,

$$ F(0) = 5.59 \pm 0.14, \quad \lambda_F = 0.08 \pm 0.02. $$

FIG. 7. $s_\pi$ dependence of form factors $F$ and $G$. 072004-10
In the latter analysis it was assumed that \( \lambda_F = \lambda_G = g'_p/g_p \) holds, which is confirmed by our analysis, albeit within large error limits.

Good agreement with the previous measurements [9] and considerably improved precision is shown in Fig. 8, where the phase shift difference \( \delta \) is plotted versus \( M_{\pi\pi} = \sqrt{s} \). A fit using Eq. (13) with relation Eq. (14), taking the central curve of the universal band with the six data points for \( \delta \) leads to the following value of the scattering length:

\[
a_0^0 = 0.229 \pm 0.015 \quad (\chi^2/\text{NDF}=4.8/5).
\]

The use of Eq. (14) then implies \( a_0^2 = -0.0363 \pm 0.0029 \).

### C. Fits to the whole data set

In this section we list the results of various fits to the whole data sample. A more detailed discussion and comparison will follow in Sec. VIII.

If we substitute the phase shifts \( \delta \) in Eq. (12) via Eq. (13) and Eq. (14) or Eq. (15) for the relation between \( a_0^0 \) and \( a_0^2 \), we can use the whole data sample in one single fit, which will yield the scattering length \( a_0^0 \), and the six form factor parameters \( f_\pi, f'_\pi, g_p, g'_p, h_p \). The remaining form factor parameters \( f_e, f_A, g_e, h_p \) have been fixed at zero. The results which are listed in Table VI are in excellent agreement with the ones derived in the previous paragraph. However, as expected, the statistical errors of the various parameters are smaller. The quality of the fit can be judged from Fig. 9. The agreement between the Monte Carlo simulation modified for the final values of the form factors and phase shifts in all five kinematic variables is very satisfactory.

In all previous fits, we have assumed that the decay rate does not depend on \( s_e \) and that there are no contributions from \( p \) waves to \( F \). To check this approximation we have allowed these form factors, one at a time, to vary in our fits too for the case where Eq. (14) was used. Table VII shows...
form factors $F$, $G$ and $H$ of the hadronic current, and their momentum dependence with a precision which has not been previously attained. We emphasize again that the analysis based on these data in six bins of invariant $\pi\pi$ mass is model independent.

The analysis which directly relates our data to the scattering length $a_0$ on the other hand, depends on additional input, which leads to slightly different results. While there is a consensus [22,55,39] on the use of the Roy equations [24] and Eq. (13) to relate the phase shifts to the scattering lengths, there exist slightly different ways of linking $a_0^0$ to $a_0^2$, and how to make use of peripheral $I=2$ data. These differences produce slightly different results for both $a_0^0$ and $a_0^2$ with overlapping statistical errors. The experimental and systematic uncertainties for both the phase shifts and scattering lengths are considerably smaller than the statistical ones and are therefore irrelevant to this discussion.

If both $a_0^0$ and $a_0^2$ are allowed to vary independently (Table VIII), we obtain a result outside the universal band in the $(a_0^0,a_0^2)$ plane, namely,

$$a_0^0 = 0.203 \pm 0.033, \quad a_0^2 = -0.055 \pm 0.023.$$  

Descotes et al. [55] have performed a fit to our published phase shifts [10], which are identical to the ones given here, and obtained

$$a_0^0 = 0.237 \pm 0.033, \quad a_0^2 = -0.0305 \pm 0.0226,$$

with a strong correlation between the two values, which we also observe in our result. Only that part of the 1$\sigma$ error contour of our result (the large ellipse in Fig. 10) which overlaps the universal band is consistent with both our and the $I=2$ data [56,57], and only within this band the solution of the Roy equations [24] used here is valid [58]. From the 1$\sigma$ contour and its central axis we may deduce how much the results listed in Table VI change if the input assumptions on the relation between $a_0^0$ and $a_0^2$ are varied. Using the lower limit of the band defined by the bracket in Eq. (14) we find a shift of $a_0^0$ by $-0.016$, while the maximum allowed upward shift inside the 1$\sigma$ contour and the band is 0.012.

Assigning these values as theoretical errors to our result, we obtain

$$a_0^0 = 0.228 \pm 0.012 \text{ stat.} \pm 0.004 \text{ syst.} \pm 0.012 \text{ theor.} \quad (25)$$

The use of Eq. (14) implies

$$a_0^2 = -0.0365 \pm 0.023 \text{ stat.} \pm 0.008 \text{ syst.} \pm 0.0031 \text{ theor.} \quad (26)$$

Since the central curve of the universal band is thought to be the best representation of the $I=2$ data, it is no surprise, that the fit of Descotes et al. [55], which used our phase shifts and those of the Geneva-Saclay experiment [9], Eq. (13) with the parametrization of Ref. [24] and the $I=2$ data below 800 MeV [56,57], gave nearly identical results

$$a_0^0 = 0.228 \pm 0.012, \quad a_0^2 = -0.0382 \pm 0.0038. \quad (27)$$

FIG. 10. (Color online) Results for the $\pi\pi$ scattering lengths $a_0^0$ and $a_0^2$ obtained from fits to the $K_{e4}$ data directly or from fits to the phase shifts obtained in this experiment. Large ellipse labeled E865: fit to our $K_{e4}$-data leaving both $a_0^0$ and $a_0^2$ as free parameters using Eq. (13) with the parameters of Ref. [24] (1$\sigma$ contour, see text for remark concerning the region outside the universal band). Medium size ellipse without label: fit of Ref. [55] (1$\sigma$ contour) to our phase shifts. Theoretical predictions: [18] (Weinberg, square), [19] [ChPT $O(p^4)$, square], and [21] [ChPT $O(p^6)$, small ellipse]. Solid curves labeled UB: universal band of allowed values based on Eq. (14). Solid curves labeled CLG: narrow band of allowed values based on Eq. (15). Solid vertical line labeled E865 ($A = 0$: analyticity constraints): fit to $K_{e4}$ data using Eq. (14) with 1$\sigma$ error limits given by dashed vertical lines. Dashed-dotted line labeled E865 ($A = 0$): analyticity and chiral symmetry constraints) fit to $K_{e4}$ data using Eq. (15) with 1$\sigma$ error limits given by dotted vertical lines.

**VIII. SUMMARY AND DISCUSSION**

The main results of this analysis are the measurements of the $\pi\pi$-phase shift difference $\delta$ near threshold and of the
This result is also shown in Fig. 8. Using the narrower band in the \((a_0^2, a_0^0)\) plane defined by Eq. (15) our result is
\[
a_0^0 = 0.216 \pm 0.013 \text{ stat.} \pm 0.004 \text{ syst.} \pm 0.002 \text{ theor.},
\]
which implies
\[
a_0^0 = -0.0454 \pm 0.0031 \text{ stat.} \pm 0.0010 \text{ syst.} \pm 0.0008 \text{ theor.},
\]
where the theoretical errors have been evaluated as before and correspond to the width of the band. Descotes et al. [55], again fitting to our phase shifts, have obtained for this case
\[
a_0^0 = 0.218 \pm 0.013, \quad a_0^0 = -0.0449 \pm 0.0033,
\]
again in agreement with our result and also with \(a_0^0 = 0.221 \pm 0.026\), obtained by Colangelo et al. [39] by direct numerical inversion of the relation between the phase shifts and the scattering lengths.

From this discussion we may deduce first that using our full data sample or the phase shifts, which we have extracted from it, in the six bins in \(M_{\pi\pi}\) leads to the same results. This will make further use of our data easy, should theoretical discussion continue and require this. Second, it has become clear that the most probable values of the two scattering lengths extracted from the \(K_{e4}\)-data and low-energy \(I = 2\) data, resting on a minimum of theroretical assumptions given by analyticity and crossing are those given in Eqs. (25) and (26), or Eq. (27). Using the additional constraints implied by chiral symmetry and the value of the scalar radius [22,39] leads to a value of the scattering length consistent within the statistical errors with this result, albeit just \(1\sigma\) lower. The authors of Ref. [55] have elaborated in detail how their ansatz differs from that of Ref. [39], and what the possible implications, if any, are for the chiral perturbation theory parameters \(\ell_3\) and \(\ell_4\) and the size of the quark condensate. In view of the large errors and also inconsistencies in the \(I = 2\) phase shift data [56,57], it seems premature to assign much significance to this minor discrepancy. Because of the reduced theoretical uncertainties we prefer to quote the values of Eqs. (28) and (29) as our final result. Both solutions for \(a_0^0\) are in very good agreement with the full two-loop standard ChPT prediction [21,22]
\[
a_0^0 = 0.220 \pm 0.005, \quad a_0^0 = -0.0444 \pm 0.0010.
\]

The influence of the reduced uncertainties of our results on the form factors \(F\), \(G\) and \(H\) on the determination of the low energy constants of ChPT is evident from recent work of Amorós et al. [27], who have updated their earlier work [26] using our data [10]. The constants \(L_1', L_2'\) and \(L_3'\) changed from 0.53±0.25, 0.71±0.27 and −2.72±1.12 (in units of \(10^{-3}\)), respectively, to 0.43±0.12, 0.73±0.12 and −2.35±0.37.

The first nonvanishing contribution to the anomalous form factor \(H\) in ChPT is predicted to be \(H = -2.67\) [59]. This agrees well with our value of \(H = −2.95±0.19\pm 0.20\). An estimation of the next to leading order gives only a small contribution [60].

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[35] The numerical values for the coefficients are listed in Ref. [24], Appendix D, and Ref. [55], Appendix B.
[45] See T.G. Trippe, in \textit{Groom} et al. [48], p. 503.
[51] See Ref. [44], Appendix 2.
[58] We thank G. Colangelo for pointing this out.