Quantum electrodynamics: Free electron

Refs: M. Stone; Bjorken and Drell.

\[ S = \int d^4x [\bar{\psi} (i\gamma^\mu \partial_\mu - m) \psi] = \int dt \, L = \int d^4x \, \mathcal{L}. \]

Varying \( \bar{\psi} \) gives \((i\gamma^\mu \partial_\mu - m)\psi = 0\) and varying \( \psi \) and integrating by parts gives \( i\partial_\mu \bar{\psi} \gamma^\mu + \bar{\psi} m = 0 \) (adjoint of same equation). The conjugate momenta are

\[ p_\psi = \delta L / \delta \partial_0 \psi = i\bar{\psi} \gamma^0 = i\psi^\dagger \quad p_{\bar{\psi}} = 0 \]

We get

\[ H = \int d^3x \, \bar{\psi} (-i\vec{\gamma} \cdot \nabla + m) \psi \]
Quantization

We require an anticommutation relation (fermions)

\[
\{\psi_\alpha(x), \psi_\beta^\dagger(x')\} = \delta_{\alpha,\beta}\delta^3(x - x').
\]  

(1)

The plane-wave expansion is

\[
\psi_\alpha(x) = \sum_r \int \frac{d^3 k}{(2\pi)^3} \frac{m}{E_k} \left[ a_{k,r}^{(r)}(k)e^{-ik\cdot x} + b_{k,r}^\dagger^{(r)}(k)e^{ik\cdot x} \right]
\]

All anticommutator combinations give 0 except

\[
\{a_{k,r}, a_{k',s}^\dagger\} = (2\pi)^3 \frac{E_k}{m} \delta_{rs} \delta^3(k - k')
\]

(2)

\[
\{b_{k,r}, b_{k',s}^\dagger\} = (2\pi)^3 \frac{E_k}{m} \delta_{rs} \delta^3(k - k')
\]

(3)

Exercise (1)

Show that (2), (3), etc imply (1) above.
Hamiltonian in terms of Fock-space operators

\[ H = \sum_r \int \frac{d^3 k}{(2\pi)^3} \frac{m}{E_k} E_k (a_{k,r}^\dagger a_{k,r} - b_{k,r} b_{k,r}^\dagger) \]

or, more generally, the total four-momentum is

\[ P^\mu = \sum_r \int \frac{d^3 k}{(2\pi)^3} \frac{m}{E_k} k^\mu (a_{k,r}^\dagger a_{k,r} - b_{k,r} b_{k,r}^\dagger) \]

Compare with our previous nonrelativistic Z-electron atom:

\[ H = \sum_{n \in \text{unoccupied}} E_n a_n^\dagger a_n + \sum_{n \in \text{occupied}} E_n b_n b_n^\dagger \]

where \( b_n^\dagger \) creates a hole of energy \(-E_n\).
Hole interpretation

So positron creation is analogous to hole creation. After normal ordering, the fully occupied atom had zero energy.

\[ H = \sum_{n \in \text{unoccupied}} E_n a_n^\dagger a_n + \sum_{n \in \text{occupied}} (-E_n) b_n^\dagger b_n \]

Hole creation then results in a loss of energy \(-E_n\). For the relativistic case after normal ordering we have

\[ P^\mu = \sum_r \int \frac{d^3 k}{(2\pi)^3} \frac{m}{E_k} k^\mu (a_{k,r}^\dagger a_{k,r} + b_{k,r}^\dagger b_{k,r}) \]

and positron creation *increases* the four momentum. In effect, we are filling a “Dirac sea” of all the negative energy states and setting the four momentum of that “ground state” to zero.
Quantum electrodynamics

With photons, now:

\[ S = \int d^4x \{ \bar{\psi} [i \gamma^\mu (\partial_\mu + ieA_\mu) - m] - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \psi \} \]

Note that \( j^\mu = e \bar{\psi} \gamma^\mu \psi \).

The Dirac equation is now

\[ [i \gamma^\mu (\partial_\mu + ieA_\mu) - m] \psi = 0 \]

Gauge invariance:

\[ \psi(x) \rightarrow \psi'(x) = e^{ie\chi(x)} \psi(x) \]
\[ A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) - \partial_\mu \chi(x) \]

Exercise (2)

Show that the action is invariant under this gauge transformation.
Quantization procedure

We treat the interaction term $-e \int d^4x \bar{\psi} \gamma^\mu \psi A_\mu$ as a perturbation and expand the electron and photon fields in terms of the free plane-wave solutions. We compute scattering amplitudes and other quantities using perturbation theory. For this we need the Feynman rules. The interaction is represented diagrammatically as shown.

Wavy lines represent photons. Solid lines, electrons. The arrows on external electron lines show the flow of electron charge. Electrons forward, positrons backward.
Momentum-space Feynman rules for QED (Feynman gauge)

1. At each order draw all distinct graphs with $n$ vertices of interest contributing to the process.
2. For each internal photon line $-ig_{\mu\nu}/(p^2 + i\epsilon)$.
3. For each internal fermion line $i(p + m)/(p^2 - m^2 + i\epsilon)$.
4. For each vertex $-ie\gamma^\mu$.
5. Enforce four-momentum conservation at each vertex.
6. For each unconstrained internal four-momentum $\int d^4p/(2\pi)^4$.
7. External incoming electron $u(p)$; outgoing $\bar{u}(p)$; incoming positron $\bar{v}(p)$; outgoing $v(p)$. Photon in or out: $\epsilon^\lambda_\mu(p)$ (polarization vector for state $\lambda$).
8. $\pm 1$ for relative interchange of fermions in final state.
9. $(-)^L$ for $L$ closed fermion loops.
Example $e^+e^- \rightarrow \mu^+\mu^-$

For our purposes the muon $\mu$ is a heavy electron.

$$\bar{\nu}^s(p')(ie\gamma^\mu)u^s(p)\left(\frac{-ig_{\mu\nu}}{q^2}\right)\bar{u}^r(k)(ie\gamma^\nu)v^{r'}(k')$$

Drop spin labels for now:

$$iT(e^-e^+ \rightarrow \mu^-\mu^+) = \frac{ie^2}{q^2} [\bar{\nu}(p')\gamma^\mu u(p)][\bar{u}(k)\gamma^\mu v(k')]$$
Scattering cross section

The cross section for an unpolarized beam and target and undetected final spins is

\[ \frac{d\sigma}{d\Omega} \propto \frac{1}{2} \sum_s \frac{1}{2} \sum_{s'} \sum_r \sum_{r'} |T|^2 = \frac{1}{4} \sum_{\text{spins}} |T|^2 \]

\[ |T|^2 = \frac{e^4}{q^4} [\bar{v}(p') \gamma^\mu u(p')]^* [\bar{v}(p') \gamma^\nu u(p')][\bar{u}(k) \gamma_\mu \nu(k')]^* [\bar{u}(k) \gamma_\nu \nu(k')] \]

Note that \( [\bar{u}(f) \Gamma u(i)]^* = \bar{u}(i) \Gamma_{\text{adj}} u(f) \). We also need

\[ \Lambda_+(p) = \sum_r u^{(r)}(p) \bar{u}^{(r)}(p) \quad \Lambda_-(p) = -\sum_r \nu^{(r)}(p) \bar{\nu}^{(r)}(p) \]
So we have

\[
\frac{1}{4} \sum_{\text{spins}} |T|^2 = \frac{e^4}{4q^2} \text{Tr} \left\{ \frac{p + m}{2m} \gamma^\mu \frac{p' - m}{2m} \gamma^\nu \right\} \times \text{Tr} \left\{ \frac{k' + M}{2M} \gamma^\mu \frac{k - M}{2M} \gamma^\nu \right\}
\]
Dirac gamma traceology

\[
\begin{align*}
\text{Tr}(1) &= 4 \\
\text{Tr}(\alpha) &= 0 \\
\text{Tr}(\alpha \beta) &= 4a \cdot b \\
\text{Tr}(\alpha \beta \gamma) &= 0 \\
\text{Tr}(\alpha \beta \gamma \delta) &= 4[(a \cdot b)(c \cdot d) + (a \cdot d)(b \cdot c) - (a \cdot c)(b \cdot d)]
\end{align*}
\]

Exercise (3)

Prove the last identity above.

So

\[
\text{Tr}\left\{ \frac{\phi + m}{2m} \gamma^\mu \frac{\phi'}{2m} - m \gamma^\nu \right\} = \frac{1}{m^2} [p^\mu p'^\nu + p^\nu p'^\mu + g^{\mu\nu} (p \cdot p' + m^2)]
\]
Scattering cross section (cont)

So

\[
\frac{1}{4} \sum_{\text{spins}} |T|^2 = \frac{e^4}{4q^4 m^2 M^2} \left[ p^\mu p'^{\nu} + p'^\nu p'^{\mu} + g^{\mu\nu}(p \cdot p' + m^2) \right]
\]

\[
\times \left[ k_\mu k'_\nu + k_\nu k'_\mu + g_{\mu\nu}(k \cdot k' + M^2) \right]
\]

\[
= \frac{e^4}{2q^4 m^2 M^2} \left[ (p \cdot k)(p' \cdot k') + (p \cdot k')(p' \cdot k) + M^2(p \cdot p') \right]
\]

where we have dropped the terms in \( m^2 \) compared with \( M^2 \), etc.
Scattering cross section (cont)

Center of mass

\[
q^2 = (p + p')^2 = (k + k')^2 = 4E^2
\]

\[
p \cdot k = p' \cdot k' = E^2 - E|k| \cos \theta
\]

\[
p \cdot k' = p' \cdot k = E^2 + E|k| \cos \theta
\]

So

\[
\frac{1}{4} \sum_{\text{spins}} |T|^2 = \frac{e^4}{32E^4m^2M^2}[E^2(E - |k| \cos \theta)^2
\]

\[+ E^2(E + |k| \cos \theta)^2 + E^2M^2]\]

\[
= \frac{e^4}{16m^2M^2} \left[ \left(1 + \frac{M^2}{E^2}\right) + \left(1 - \frac{M^2}{E^2}\right) \cos^2 \theta \right]
\]
Scattering cross section (cont)

\[
\frac{d\sigma}{d\Omega} = \frac{1}{2E_{\text{cm}}^2} \frac{|k|m^2M^2}{\pi^2E_{\text{cm}}} \frac{1}{4} \sum_{\text{spins}} |T|^2 \\
= \frac{\alpha^2}{4E_{\text{cm}}^2} \sqrt{1 - \frac{M^2}{E^2}} \left[ \left(1 + \frac{M^2}{E^2}\right) + \left(1 - \frac{M^2}{E^2}\right) \cos^2 \theta \right] \\
= \frac{\alpha^2}{4E_{\text{cm}}^2} \beta \left[1 + \cos^2 \theta + (1 - \beta^2) \sin^2 \theta \right]
\]

where \( \beta = v/c = \sqrt{1 - M^2/E_{\text{cm}}^2} \) for the muon, and \( \alpha = e^2/(4\pi) \). At high energy \( E = E_{\text{cm}}/2 \gg M \):

\[
\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_{\text{cm}}^2} \left(1 + \cos^2 \theta\right)
\]
The initial interaction is modeled as simple pair production. The quark and antiquark separate and interact by means of gluon exchange and further quark-antiquark production, resulting, finally, in collimated back-to-back “jets” of hadrons. We calculate the ratio

$$R(s) = \frac{\sigma(e^+e^- \rightarrow \text{hadrons})}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)} \approx \sum_i N_i Q_i^2$$

where $Q_i$ are the charges of the quarks in units of the electron charge and $N_i$ counts the number of quarks of charge $Q_i$. (Each quark comes in three color states, so $N_i = 3$.) The sum runs over the quarks that have masses less than $\sqrt{s}/2$. 
### R (theory)

<table>
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<th>quark</th>
<th>$Q_i$</th>
<th>mass (MeV)</th>
<th>$N_i Q_i^2$</th>
<th>$\sum_i$</th>
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<td>$&lt; 10$</td>
<td>$4/3$</td>
<td>$1.3$</td>
</tr>
<tr>
<td>d</td>
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<td>$&lt; 10$</td>
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</table>

So $R \approx 2$ up to $2 \times 1.3$ GeV and $3.3$ up to $2 \times 4.4$ GeV. There is also some contribution from the $\tau$ lepton (like the $\mu$, but with mass $1780$ MeV). It contributes $f N_i Q_i^2 \approx 0.8$ above about $3.5$ GeV. The fraction $f \approx 0.8$ represents the fraction of $\tau$ decays resulting in hadrons.
From Ezhela, Lugovsky, and Zenin arXiv:hep-ph/0313114. The broken green curve is the prediction we have calculated. The peaks show a variety of resonances.