1. Consider relativistic motion of a particle with the charge $q$ and the mass $m$ in homogeneous electric field $E$. Choose $z$ as the field direction. Initial conditions are: $p_z = p_y = 0$ and $p_x = p_0$. Using equation of motion

$$\frac{dp}{dt} = qE,$$

find

a. all components of velocity $v$ as a function of time. Find limits of these components at $t \to \infty$ and explain these limits (5 points).

b. Find equation of the trajectory $z(x)$ in the plane $zx$ that goes through the point $x = 0, z = H(0)/qE$, where $H(t)$ is kinetic energy of the particle (4 points).

Hint: Use $v = pc^2/H$, where $H = c\sqrt{(m^2c^2 + p^2)}$.

The equation of motion reads

$$\frac{dp}{dt} = qE.$$ (2)

Then at $t > 0$ one gets $p_z = qEt, p_x = p_0, p_y = 0$.

The kinetic energy of the particle

$$H(t) = c\sqrt{m^2c^2 + p^2} = \sqrt{H(0)^2 + (cEqt)^2},$$ (3)

with $H(0)^2 = m^2c^4 + c^2p_0^2$.

Velocity $v(t) = pc^2/H$. Then

$$\frac{dz}{dt} = v_z(t) = \frac{c^2qEt}{\sqrt{H(0)^2 + (cEqt)^2}},$$ (4)

and

$$\frac{dx}{dt} = v_x(t) = \frac{c^2p_0}{\sqrt{H(0)^2 + (cEqt)^2}}.$$(5)

and $v_y = 0$.

$\lim_{t \to \infty} v_z(t) = c$ and $\lim_{t \to \infty} v_x(t) = 0$.

Explanation: $v_z \to c$ because the particle is accelerated by field in $z$-direction, but the velocity cannot be larger than $c$.

$v_x \to 0$ because $p_x = p_0 = mv_x/\sqrt{1 - v^2/c^2}$. Since $p_0$ is time independent and $\sqrt{1 - v^2/c^2} \to 0$, one gets $v_x \to 0$. 

Integrating Eq. (4) one gets
\[ z = \sqrt{H(0)^2 + (cEqt)^2/qE}. \] (6)

Integrating Eq. (5) one gets
\[ x = \left(\frac{p_0c}{qE}\right) \sinh^{-1}\left(cqEt/H(0)\right) \] (7)

Excluding \( t \) from Eq.(6) and Eq(7) one gets
\[ z = \left(\frac{H(0)/qE}{c}\right) \cosh(qEx/p_0c) \] (8)

2. Find three possible gauges of vector-potential \( \mathbf{A} \) for the case when \( \mathbf{B} = \text{curl} \mathbf{A} \) is a homogenous and time-independent field in z-direction.

\( A_y = A_z = 0, A_x = -By \)
\( A_x = A_z = 0, A_y = Bx \)
\( \mathbf{A} = \left|\mathbf{B} \times \mathbf{r}\right|/2. \)

3. Dynamics of relativistic particles.
A particle of mass \( m_1 \) with velocity \( v_1 \) hits a particle at rest with the mass \( m_2 \) and they stick together. Find velocity \( V \) and mass \( M \) of the compound particle.

The momentum of the system before and after the collision is
\[ \vec{p} = \frac{m_1v_1^2}{\sqrt{1 - v_1^2/c^2}}. \] (9)

The energy of the system \( E \) before and after the collision is
\[ E = \frac{m_1c^2}{\sqrt{1 - v_1^2/c^2}} + m_2c^2 \] (10)

To find \( \vec{V} \) one may use equation for the compound particle
\[ \vec{p} = E\vec{V}/c^2 \] (11)

To find \( M \) one can use
\[ E = c\sqrt{p^2 + M^2c^2} \] (12)

The results are
\[ \vec{V} = v_1 \frac{m_1}{m_1 + m_2\sqrt{1 - v_1^2/c^2}} \] (13)

and
\[ M^2 = m_1^2 + m_2^2 + \frac{2m_1m_2}{\sqrt{1 - v_1^2/c^2}}. \] (14)