Determining the Acceleration Due to Gravity with a Simple Pendulum

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Using a simple pendulum the acceleration due to gravity in Salt Lake City, Utah, USA was found to be \( (9.8 \pm 0.1) \text{ m/s}^2 \). The model was constructed with the square of the period of oscillations in the small angle approximation being proportional to the length of the pendulum. The model was supported by the data using a linear fit with chi-squared value: 0.77429 and an r-square value: 0.99988. This experimental value for gravity agrees well with and is within one standard deviation of the accepted value for this location.

I. INTRODUCTION

The study of the motion of the simple pendulum provided valuable insights into the gravitational force acting on the students at the University of Utah. The experiment was of value since the gravitational force is one all people continuously experience and the collection and analysis of data proved to be a rewarding learning experience in error analysis. Furthermore, this experiment tested a mathematical model for the value of gravity that that makes use of the small-angle approximation and the proportional relationship between the square of the period of oscillations to the length of the pendulum. Sources of error for this procedure included precision in both length and time measurement tools, reaction time of the stopwatch holder, and the accuracy of the stopwatch with respect to the lab atomic clock. The final result of \( g \) takes into account the correction for the error introduced using the approximation. There are opportunities to correct for the effects of mass distribution, air buoyancy and damping, and string stretching[1]. Our results do not take these effects into account at this time.

A. Theoretical Introduction

The general form of Newton’s Law of Universal Gravitation can be used to find the force between any two bodies.

\[
\vec{F}_G = -G \frac{mM_E}{R_E^2} \vec{r}
\]  

(1)
On earth this equation can be simplified to \( F = -mg \dot{r} \) with the value \( \frac{GM_E}{R_E^2} \) taken to be the constant \( g \). The value of gravity in Salt Lake City (elev. 1320 m) according to this model is: 9.81792 m/s\(^2\)[3][4][5].

The simple pendulum provides a way to repeatedly measure the value of \( g \). The equation of motion from the free body diagram in Figure 1[2]:

\[
F = ma = mgsin\theta
\]  

(2)

\[
\ddot{\theta} - \frac{g}{L} \theta = 0
\]  

(3)

The solution to this differential equation relies on the small angle approximation \( sin\theta \simeq \theta \) for small \( \theta \):

\[
\theta(t) = \theta_0 cos\left(\sqrt{\frac{g}{L}}t\right)
\]  

(4)

FIG. 1: Free body diagram of simple pendulum motion[2].
The Taylor expansion

\[
\theta(t) \simeq \theta_0 \left[ 1 - \frac{gt^2}{2L} + \frac{g^2t^2}{4!L^2} \right]
\]  

allows us to take the \( \theta \) dependence out of the equation of motion. Taking the second derivative of the approximation gives the following:

\[
\ddot{\theta} = -\theta_0 \frac{g}{L}
\]  

(6)

\[ -\theta_0 \frac{g}{L} + \theta_0 \frac{g}{L} = 0 \implies \theta_0 \frac{g}{L} = \theta \frac{g}{L} \]  

(7)

We know from the first derivative \( \dot{\theta} = \omega = \sqrt{\frac{g}{L}} \), so it follows that since \( \omega^2 = \frac{4\pi^2}{T^2} = \frac{g}{L} \)

\[
\theta_0 \frac{g}{L} = \theta \frac{4\pi^2}{T^2}
\]  

(8)

From the initial conditions it is also clear that the initial amplitude \( \theta \) is equal to \( \theta_0 \) and so the linear relationship between length \( L \) and period \( T^2 \) can be expressed as

\[
T^2 = \frac{4\pi^2}{g}L
\]  

(9)

Using the small angle approximation introduces a small systematic error in the period of oscillation, \( T \). For instance the maximum amplitude angle \( \theta \) for a 1 percent error is .398 radians or 22.8 degrees; to reduce the error to 0.1 percent the angle must be reduced to .126 radians or 7.2 degrees. This experiment used an angle of about 10 degrees and that introduced an error of 0.3 percent. The calculations for the systematic error are found in the Appendix.

II. EXPERIMENTAL PROCEDURE

A. Setup

As seen in Figure 2, the pendulum apparatus was set up using a round metal bob with a hook attached to a string. The string passed through a hole in an aluminum bar, which was attached to
the wall. The length of the string could be adjusted, and the precise point of oscillation was fixed by a screw, which also connected a protractor to the aluminum bar.

![Experiment setup](image)

**FIG. 2: Experiment setup.**

Length measurements for the pendulum were taken using a meter stick and caliper. The caliper was used to measure the diameter of the bob, having an uncertainty of 0.01 cm. The total length was measured by holding the meter stick up against the aluminum bar, and measuring from the pivot point to the bottom of the bob. The bottom was determined by holding a ruler horizontally against the bottom of the bob. The meter stick measurements had an uncertainty of 0.2 cm.

Time measurements were made using a stopwatch. For measuring the first swing the starting time was determined by holding the bob in one hand and the stopwatch in the other and simultaneously releasing the bob and pushing Start. The stopping point, and starting point for the second oscillation, was determined by watching the bob and pushing Stop/Start when the bob appeared to reach the top of the swing and stop. The precision of the stopwatch was compared with an atomic clock by measuring several one second intervals. The precision of the time measurements were also affected by reaction time and perception of starting and stopping points of the person taking the measurements. Time measurements were taken by the same person to keep the uncertainty in reaction time consistent.
B. Procedure

To determine which measurements were most reliable, data was taken for the period of the first oscillation, second oscillation, and twenty oscillations (omitting the first) at a set length of 20.098 cm. The length was then adjusted to 65.5647 cm, and the same measurements were taken. To see the limits of the small angle approximation measurements of 20 oscillations (omitting the first) at a fixed length of 60.1605 cm were taken by beginning the swing at angles of 5, 10, 20, and 40 degrees. Measurements were then taken for 20 oscillations (omitting the first) for lengths of 20.098, 26.898, 32.898, 60.1605, 65.56467, 74.648, 89.848, 104.548, 116.498, and 129.898 cm at a starting angle of about 10 degrees.

III. RESULTS

The result for \( g \) obtained from both measured values of \( L \) and \( T^2 \) from equation 9 as well as from the slope in the Linear Fit model (Figure 4) agree very well with accepted results for \( g \). The precision could be improved by corrections for effects of mass distribution, air buoyancy and damping, and string stretching[1].

<table>
<thead>
<tr>
<th>Degrees</th>
<th>3</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average Period of 20 Oscillations</td>
<td>31.18333</td>
<td>31.24833</td>
<td>31.266</td>
<td>31.50833</td>
<td>32.06667</td>
</tr>
<tr>
<td>Average Period of Oscillation</td>
<td>1.559167</td>
<td>1.562417</td>
<td>1.5633</td>
<td>1.575417</td>
<td>1.60333</td>
</tr>
</tbody>
</table>

IV. DISCUSSION

By measuring 20 oscillations the average period is determined by dividing by 20 and this helps reduce the error since the error propagation will provide an uncertainty in the period that is the uncertainty in the time measurement divided by twenty.

From Table 1 and Figure 3 the limits of the small angle approximation are shown. Between 10 and 20 degrees the theoretical model begins to breakdown and the measured period deviates from the theoretical value. Measurements taken at less than 10 degrees will be more accurate for the small angle approximation model that was used.

Two methods were used to calculate a value of \( g \) from the data. The first method used to calculate a value of \( g \) from the measurements taken is making the calculation from each of the
ten different lengths, using the measurements shown in Table 7 of 20 oscillations at the different lengths, and taking the average. The calculated average $g$ was $(9.7 + / - 0.1)$ m/s$^2$.

The second method used was applying a linear least squares fit to the values of length and the
accompanying $T^2$. Figure 4 shows this method and gives the values for the fit parameters. The value of $g$ is determined by using the slope of the line and gave a value of $g$ to be $(9.8 + / - 0.1)$ m/s$^2$.

Figure 5 shows that data has a random pattern and all of the error bars go through zero, which means that the data is a good fit for a linear model.

![FIG. 5: Random pattern of Residual $T^2$.](image)

As discussed in the theoretical introduction, a value of $g$ $9.81792$ m/s$^2$ can be calculated using $G$, $M_E$, and $R_E$. The value of $g$ varies depending on location due to several factors including the non-sphericity of the earth, and varying density. A more accurate value of $g$ in Salt Lake City, Utah can be calculated by taking into account these effects.

The National Geodetic Survey website, which interpolates the value of $g$ at a specific latitude, longitude and elevation from observed gravity data in the National Geodetic Survey’s Integrated Data Base, was used to determine an accepted value of $g$ for Salt Lake City, Utah, for which to compare the calculated results\[7]\[8]\[6]. The accepted value for $g$ in Salt Lake City, Utah is $(9.79787 + / - 0.00002)$ m/s$^2$.

Comparing the two methods used to calculate $g$ shows that the least squares linear fit provided a value of $g$ that is closer to the theoretical\[3]\[4]\[5] and accepted\[7]\[8]\[6] values of $g$.

The calculation of $g$ supports the small angle approximation model that was used. The linear relationship to length and period squared provided by the approximation gave a way of employing a least squares linear fit to the data to determine a value of $g$. Since the calculated value was
within one standard deviation from the theoretical value, the model was supported.

V. CONCLUSION

The small angle approximation model, which gives $g$ as being proportional to $T^2$ and $L$, was supported by the data taken using a simple pendulum. The residual of the data showed that it was a good fit for a linear model, and the least squares linear fit of the data had fit parameters of chi-squared: 0.77429 and an r-square value: 0.99988. The value of $g$ taken from the slope of the least squares linear fit provided a value of $g$: $(9.8 + / - 0.1) \text{ } m/s^2$, which is within one standard deviation of the accepted value of gravity in Salt Lake City: $9.79787 \text{ } m/s^2$. The experiment was a good way of testing the small angle approximation because the period measured using different starting angles was consistent for angles less than 10 degrees. Using the small angle approximation the relationship between period squared and length was linear so a least squares linear fit could be utilized to calculate $g$. The value of $g$ calculated using the least squares linear fit could then be compared to the accepted value of $g$ for the location, thus verifying the model that was employed.

VI. APPENDIX A

A. Error Analysis

B. Time

The sources of error introduced in this experiment came from the tools we used to measure length: calipers for the bob and a meter stick for the string length as well as the stop watch used to time each period of oscillation. Measuring the period had several sources of error including precision, the atomic clock benchmark, the reaction time of the experimenter, and the statistical error which was the standard deviation from the measurements taken. On the whole, the relative error in $T$ was greater so that was the error used in the linear fit analysis.

$$\delta T = \frac{1}{20}\sqrt{(\delta T_{\text{reaction}})^2 + (\delta T_{\text{atomic}})^2 + (\delta T_{\text{precision}})^2 + (\delta T_{\text{statistical}})^2} \tag{10}$$

Equation 10 also takes into account the error propagation in taking the time period for twenty oscillations. This $\delta T$ is the random error; to account for the systematic error introduced by using the small angle approximation the complete solution for the period of oscillation is as follows [2]:

$$T(\theta_{\text{max}}) = T_0 + T_0\left[\frac{1}{4}\sin^2\left(\frac{\theta_{\text{max}}}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_{\text{max}}}{2}\right)\right] \tag{11}$$

To find the percent error introduced by the angle used in the experiment the solution in equation 11 was rearranged to give:

$$\frac{T(\theta_{\text{max}}) - T_0}{T_0} = \frac{1}{4}\sin^2\left(\frac{\theta_{\text{max}}}{2}\right) + \frac{9}{64}\sin^4\left(\frac{\theta_{\text{max}}}{2}\right) \tag{12}$$

The angle used in this experiment was 10 degrees. Plugging that value into the right side of equation twelve gives a value of $0.002967$. It follows that

$$T_0 = \frac{T(\theta_{\text{max}})}{1.002967} \tag{13}$$

Each of our measured values of $T$ was corrected by this factor. To get the error for $T^2$:

$$\delta T^2 = T\delta T \tag{14}$$

The results are found in Table 7. These values were plotted in figures 4 and 5.
C. Gravity

The errors in the calculations for \( g \) were determined differently for the two methods. The uncertainty in the least square fit was calculated from the slope and uncertainty of the slope (see Figure 4).

\[
\delta g = \frac{4\pi^2}{m^2} \delta m \tag{15}
\]

The calculations of \( g \) from \( L \) and \( T^2 \) used:

\[
\delta g = g \sqrt{\left( \frac{\delta L}{L} \right)^2 + \left( 2 \frac{\delta T^2}{T} \right)^2} \tag{16}
\]

These values are found in Table 8.