**Introduction to Numerical Analysis**

In this lesson you will be taken through a pair of techniques that will be used to solve the equations of motion

\[ v = \frac{dx}{dt} \]

and

\[ a = \frac{F}{m} = \frac{dv}{dt} \]

for situations in which \( F \) is well known, and the initial conditions are stated. In the lesson you will find the solution to the above equations for a skydiver using the numerical technique called the Euler (pronounced “oiler”) or Euler-Cauchy method. In the lab that follows you will be provided with an Excel spreadsheet that performs such calculations for a harmonic oscillator and you will be able to compare the numerical results with experimental results.

Many times--particularly in introductory courses--the complete exploration of a topic in physics is halted when the theory leads up to a differential equation. The detailed solution of the equation would yield valuable results, and provide new insights into the concept being covered, but is usually beyond the scope of the course. Often a limiting simplifying assumption is introduced, which allows the solution to proceed in a form that is easier to handle. These simplifying assumptions are often quite sufficient. This lab gives you something to try when they are not.

A variety of easy to use methods exist to handle the solution of differential equations, called numerical methods. These methods involve "chopping up" a problem into many "bite-size" parts that are easy to analyze. Simple computational techniques are then applied to each of these parts. The result is that a very nearly exact solution to the equation, built out of many simple calculations.

The remainder of this lesson will deal with: (1) a description of the Euler method, and (2) the use of this method to solve the problem of finding the motion of a skydiver without neglecting air resistance.

**THE EULER METHOD**

Notation specific to one dimensional motion will be used here instead of general mathematical notation found in most texts on numerical analysis, in order to smooth the transition to solving the physics problem with unfamiliar mathematics. The problem starts with a differential equation whose solution is sought. The equation is nothing more than a statement of Newton's Second Law.

\[ a = \frac{F_{\text{net}}}{m} = \frac{dv}{dt} \]

In the skydiver case, two major forces work on such an object: gravity pulling straight down, and friction, or air resistance, retarding the fall. Actually, there is a third force acting upward that is omitted because it is small compared to the other two forces. What is it?
Forces Acting on the Skydiver

Typically this friction force can be approximated as being proportional to the square of the skydiver's speed. The total force acting on the object can then be written:

\[ F_{\text{net}} = mg - kv^2 \]

In this case \( k \) is a constant that describes the air resistance, with units of kg/m. As the object begins falling, with zero initial velocity, it begins accelerating downward at 9.8 m/s\(^2\). As the skydiver gains speed, the upward viscous damping force begins acting to decrease the downward acceleration, until the body reaches an equilibrium state, or terminal velocity. At this point the net force acting on it is zero, since the acceleration is zero. Thus:

\[ mg = kv_{\text{term}}^2 \]

Thus, the terminal velocity is easily found to be:

\[ v_{\text{term}} = \sqrt{\frac{mg}{k}} \]

If we assumed that the skydiver reached terminal velocity very quickly, then the problem is quite easy to solve. However, the exact solution of the skydiver's motion, which is far more interesting, is rather difficult. The application of Euler's method to this motion, however, allows us to find this solution numerically, so that we can study the behavior of this system in detail. We can watch the object approach equilibrium, see the effects of varying the value of \( k \), or see the effect of assuming different models for the damping force (i.e. have it depend on \( v \) or \( v^3 \)).

From the above equation for \( F_{\text{net}} \) we have:

\[ \frac{dv}{dt} = g - \frac{k}{m} v^2 \]

where \( g \) = acceleration due to gravity, \( k \) = frictional damping constant, and \( m \) = mass of falling body.

The numerical approach will take the entire time interval \( t \) during which the motion is studied and divide it into small (usually very small) intervals \( \Delta t \). The problem then amounts to producing a solution at a succession of moments in time, where each moment is separated from the next by \( \Delta t \). To begin, initial values are required, i.e., values for \( v_0 \), \( x_0 \), and \( a_0 \) (or equivalently \( F_0/m \)) at \( t = 0 \). In our case,
\[
\frac{dv_0}{dt} = a_0 = g - \frac{k}{m} v_0^2
\]

Notice that if \(v_0 = 0\), then \(a_0 = g\), as expected.

Over the \(\Delta t\) seconds of the first time interval, we pretend that \(a\) and \(v\) are constant. Thus, at \(t_i = 0 + \Delta t\):

\[
v_1 = v_0 + a_0 t = v_0 + \left(g - \frac{k}{m} v_0^2\right) \Delta t
\]

\[
x_1 = x_0 + v_0 \Delta t
\]

\[
a_1 = \frac{F_1}{m} = g - \frac{k}{m} v_1^2
\]

As long as the initial conditions are specified, \(a_i\), \(v_i\), and \(x_i\) are all easily calculated.

At the end of the second interval, when \(t_2 = t_1 + \Delta t\), we have

\[
v_2 = v_1 + a_1 \Delta t
\]

\[
x_2 = x_1 + v_1 \Delta t
\]

\[
a_2 = g - \frac{k}{m} v_2^2
\]

Again, the quantities \(v_2\), \(x_2\), and \(a_2\) are all easily computed, since \(v_1\), \(x_1\), and \(a_1\) have already been evaluated. As you can see, the solution proceeds step by step over intervals in which the values of the velocity, position, and acceleration at the beginning of the interval serve as constant inputs to calculate those same quantities at the end of the interval. Thus, the smaller the interval, the better the approximation will be. The process then repeats itself, revealing the motion of the falling skydiver. The following figure displays the approach for a general case in which the force law has not yet been stated.
Let's look at the solution in more detail. Start at the initial values of all quantities which are either known, or as in the case of \( F_0 \), easily computed.

\[
\begin{align*}
  x_0 & \equiv x \\
  v_0 & \equiv \dot{x} \\
  a_0 & = \frac{F_0}{m} = g - \frac{k}{m} v_0^2 \\
  F_0 & = F(v_0, x_0, t_0) = mg - kv_0^2
\end{align*}
\]

At the end of the first interval,

\[
\begin{align*}
  t & = t_1 = \Delta t \\
  x_1 & \equiv x_0 + v_0 \Delta t \\
  v_1 & \equiv v_0 + a_0 \Delta t \\
  a_1 & \equiv \frac{F_1}{m} \\
  F_1 & \equiv F(v_1, x_1, t_1) = mg - kv_1^2
\end{align*}
\]

At the end of the second interval,

\[
\begin{align*}
  t & = t_2 = 2\Delta t \\
  x_2 & \equiv x_1 + v_1 \Delta t \\
  v_2 & \equiv v_1 + a_1 \Delta t \\
  a_2 & \equiv F_2/m \\
  F_2 & = F(v_2, x_2, 2\Delta t) \\
  a_n & = F_n/m
\end{align*}
\]
\( t_2 = t_1 + \Delta t = 2\Delta t \)
\( x_2 \approx x_1 + v_1\Delta t \)
\( v_2 \approx v_1 + a_1\Delta t \)
\( a_2 \approx \frac{F_2}{m} \)
\( F_2 \approx F(x_2, v_2, t_2) = mg - kv_2^2 \)

These equations can be written more generally. For the \( n^{th} \) interval, we have:

\( t_n = t_{n-1} + \Delta t = n\Delta t \)
\( x_n \approx x_{n-1} + v_{n-1}\Delta t \)
\( v_n \approx v_{n-1} + a_{n-1}\Delta t \)
\( a_n \approx \frac{F_n}{m} \)
\( F_n \approx F(x_n, v_n, t_n) = mg - kv_n^2 \)

Using this repeating procedure it is possible to get an approximate picture of the motion, without actually solving the differential equation. The accuracy of this method depends on the size of \( \Delta t \); better accuracy generally being achieved as \( \Delta t \) is made smaller.

Note that such repetitive computations, although each simple in and of itself, represent quite a tedious arithmetic barrage when seen together. Here's where a spreadsheet, like Excel, is ideal. It can be programmed \textbf{both} with the numerical method \textbf{and} the force law and carry out the evaluations necessary at each value of \( \Delta t \). You can then rapidly plot \( x \) vs. \( t \) or \( v \) vs. \( t \) and get a quick picture of the motion.

The method we have outlined above is called the Euler Method, also called a "full increment method" or "tangent line" method of solution. In general, this method produces only approximate results, and very small \( \Delta t \) values are required. Hence, there are many iterations to achieve reasonable accuracy. Details on how to estimate the errors encountered in a Euler's method solution are covered in many texts on differential equations or numerical methods.
SOME MATHEMATICAL DETAILS

The nature of the approximate solution can be seen by starting with the original differential equation:

\[ a = \frac{dv}{dt} \]

where we seek values of \( v \) at time \( t \) and all later times. The solution is generated by expanding \( v \) in a Taylor series:

\[
v(t + \Delta t) = v(t) + \left( \frac{dv}{dt} \right) \Delta t + \left( \frac{d^2v}{dt^2} \right) \frac{(\Delta t)^2}{2!} + \left( \frac{d^3v}{dt^3} \right) \frac{(\Delta t)^3}{3!} + \cdots
\]

The derivatives are evaluated at time \( t \) and \( \Delta t \) is assumed small. When \( \Delta t \) is small, terms in higher orders of \( \Delta t \) become negligible. In the Euler method only the first term in \( \Delta t \) is retained and the majority of the error, then, is contained in the second order term, \((\Delta t)^2\). Thus, the Euler method is called a first order method, and the major error term with the \((\Delta t)^2\) is called the truncation error.

Clearly the smaller \( \Delta t \) is the closer the calculated \( v(t) \) will be to the actual value the skydiver has. Of course, the smaller \( \Delta t \) is the more calculations are required to produce a complete picture of the skydiver's motion.

Additionally, large increments of \( \Delta t \) will be more apt to give poor results for functions that change rapidly, even though the larger increments will save computer time and memory. For example, if the function oscillates several times during the interval \( \Delta t \), then the approximation will miss the oscillations and give a slowly varying function as a solution. Thus, the Euler method is not ideal for rapidly varying functions, especially when sufficiently small \( \Delta t \) intervals are not practical.

EXERCISES

Use a spreadsheet program to solve the following problems. You should turn in a data table and plot for position, velocity, and acceleration for each situation.

1. Solve the skydiver problem under the following conditions:

   vertical free fall (no horizontal components of velocity), \( v_o = 0 \) m/s, \( k = 0.25 \) kg/m, \( m = 70 \) kg, \( x_o = 0 \) m, \( t_o = 0 \) s and \( t_f = 15.0 \) s

   Produce solutions for: \( \Delta t = 1.0 \) s, \( \Delta t = 0.5 \) s, and \( \Delta t = 0.2 \) s.

2. Using \( \Delta t = 0.5 \) s, change the value of \( k \) or \( m \) and compare to the original solution.
QUESTIONS

Q1. For each $\Delta t$ interval in problem 1, at what value of $t$ was terminal velocity, $v_T$, reached, if it was reached, and what were the values of $v_T$?

Q2. How do you suppose the terminal velocity (from Q1) would change if $m$ was reduced to 10 kg, while $k$ is left the same?

Q3. How do you suppose the terminal velocity (from Q1) would change if the force law were $F_{net} = mg - kv^3$? How would the terminal velocity change if $F_{net} = mg - kv$?