A) Superposition of waves of same \( \lambda \), originating from 2 point sources \( S_1 \) and \( S_2 \). Separation \( a \) of sources much larger than \( \lambda \); \( P \) far away so that wavefronts at \( P \) will be planar. Assume that waves are linearly polarized:

\[
\hat{E}_1 = \hat{E}_{01} \cos (k_1 \hat{r} - \omega t + \gamma_{01})
\]

\[
\hat{E}_2 = \hat{E}_{02} \cos (k_2 \hat{r} - \omega t + \gamma_{02})
\]

**Question:** intensity of light at point \( P \)?

**Answer:**

\[
I = \mathcal{E} V \left< \hat{E}^2 \right>
\]

\[
= \mathcal{E} V \left< (\hat{E}_1 + \hat{E}_2)^2 \right> = \mathcal{E} V \left< \hat{E}_1^2 + \hat{E}_2^2 + 2 \hat{E}_1 \hat{E}_2 \right>
\]

**Interference term:**

\[
\hat{E}_1 \hat{E}_2 = \hat{E}_{01} \cos (k_1 \hat{r} - \omega t + \gamma_{01}) \times \hat{E}_{02} \cos (k_2 \hat{r} - \omega t + \gamma_{02})
\]

\[
= \hat{E}_{01} \hat{E}_{02} \left[ \cos (k_1 \hat{r} + \gamma_{01}) \cos \omega t + \sin (k_1 \hat{r} + \gamma_{01}) \sin \omega t \right]
\]

\[
\times \left[ \cos (k_2 \hat{r} + \gamma_{02}) \cos \omega t + \sin (k_2 \hat{r} + \gamma_{02}) \sin \omega t \right]
\]
\[ \langle \cos^2 \omega t \rangle = \frac{1}{2} \quad \text{and} \quad \langle \sin^2 \omega t \rangle = \frac{1}{2} \quad \text{and} \quad \langle \cos \omega t \sin \omega t \rangle = 0 \]

\[ \langle \vec{E}, \vec{E}_2 \rangle = \frac{1}{2} \cdot \bar{E}_0 \cdot \bar{E}_2 \cdot \cos \left( \frac{\vec{k}_1 \cdot \vec{r} + \vec{y}_1 - \vec{k}_2 \cdot \vec{r} - \vec{y}_2}{\lambda} \right) \]

\[ \equiv \text{phase difference} \delta, \text{arising from combined path length and initial phase angle difference} \]

1) if \( \vec{E}_0, \vec{E}_2 \) perpendicular : \( \langle \vec{E}, \vec{E}_2 \rangle = 0 \)

\[ I = I_1 + I_2 \]

2) most important situation : \( \vec{E}_0, \vec{E}_2 \) parallel

\[ I = I_1 + \frac{I_2}{E_0^2} + \frac{2 \sqrt{I_1 I_2}}{E_1 E_2} \cdot \cos \delta \]

\[ \equiv \frac{2\pi}{\lambda} \cdot \Delta x \]

depending on interference term, resulting intensity can be larger, equal, or smaller than \( I_1 + I_2 \):

a) \( \cos \delta = 1 \) (case for \( \Delta x = \) multiple of \( \lambda \), or \( \delta = \text{multiple of} \ 2\pi \) i.e. \( \delta = \pm 2\pi, \pm 4\pi \))

\[ I_{\text{max}} = I_1 + I_2 + 2 \sqrt{I_1 I_2} \quad \text{total constructive interference} \]

b) \( 0 < \cos \delta < 1 \) (waves are out of phase)

\[ I_1 + I_2 < I < I_{\text{max}} \quad \text{constructive interference} \]
c) \( 0 > \cos \delta > -1 \)
\[
I_1 + I_2 > I > I_{\text{min}} \quad \text{destructive interference}
\]

d) \( \cos \delta = -1 \) \( (\delta = \pm \pi, \pm 3\pi, \pm 5\pi) \)
waves are 180° out of phase, throughs overlap crests
\[
I_{\text{min}} = I_1 + I_2 - 2\sqrt{I_1 I_2} \quad \text{"total destructive interference"}
\]

important special cases if \( I_1 = I_2 \):
\[
\Rightarrow I = 2I_0 \left(1 + \cos \delta\right) = 4I_0 \cos^2 \frac{\delta}{2}
\]
\[
I_{\text{min}} = 0 \quad \text{optimum contrast in interference pattern}
\]
\[
I_{\text{max}} = 4I_0
\]

conditions for interference:

i) phase difference must be constant \( \Rightarrow \) coherent waves

ii) nearly same frequency

\[
\text{otherwise} \quad \langle I_{12} \rangle = 0
\]

iii) clearest interference patterns for interfering wave if interfering waves have same intensity

iv) two orthogonally polarized beams cannot interfere.
Superposition holds for any number of constituents

a) very large number of independent light sources (N) with random phase shifts.

Interference term \( \cos \delta = \cos \frac{2\pi}{\lambda} \Delta x \) takes on both positive and negative values with some likelihood:

cancels out.

\[
E_0^2 \text{ total} = \sum_{i=1}^{N} E_{0i}^2 = NE_{01}^2
\]

(\text{flux density } \sim E_0^2)

Total intensity of N sources with random phases given by sum of individual intensities.

b) all N sources are "coherent" and in phase (\(\Delta x = 0\))

\[
\begin{align*}
E_0^2 \text{ total} &= E_{01}^2 + E_{02}^2 + \ldots E_{0i}^2 + \sum_{i<j} E_{0i} E_{0j} \\
E_0^2 \text{ total} &= \left( \sum_{i=1}^{N} E_{0i} \right)^2 = \left( NE_{01} \right)^2 = N^2 E_{01}^2
\end{align*}
\]

\(\uparrow\)

if all \(E_{0i}\) are amplitudes add \(\uparrow\) the same
3) Two travelling waves of same frequency and amplitude propagating in opposite direction

\[ E_{\text{total}} = E_0 \left[ \sin \frac{2\pi}{\lambda} \left( \frac{x}{\lambda} - vt \right) + \sin \frac{2\pi}{\lambda} \left( \frac{x}{\lambda} + vt \right) \right] \]

\[ \sin \alpha + \sin \beta = 2 \sin \frac{1}{2} (\alpha + \beta) \cdot \cos \frac{1}{2} (\alpha - \beta) \]

\[ = 2 E_0 \sin \frac{2\pi}{\lambda} \cdot \cos \frac{2\pi}{\lambda} \cdot \cos vt \]

profile does not move through space \( j \) at each point \( x = 0, \frac{\lambda}{2}, \frac{3\lambda}{2}, \ldots \quad E = 0 \), all the time.

These points are called nodes. Halfway between nodes, at \( x = \frac{\lambda}{4}, \frac{3\lambda}{4}, \ldots \) maximum amplitude

\[ 2 E_0 \] varies periodically with \( \cos vt \).

Standing waves of light first observed by W. B. Wood.

Light reflected from plane mirror. Thin, transparent photographic film \( (d < \lambda/20) \) on glass inclined to mirror by angle \( \approx 10^{-3} \) degrees; casts across pattern of standing waves.
Develop film \( \Rightarrow \) equidistant stripes show "anti-nodes" of electric field (magnetic nodes and antinodes in standing wave alternate with those of the electric field) \( \Rightarrow \) Electric field triggers photographic process.

C. Combination of Waves with different frequencies

\( \nu_1 \) ; \( \nu_2 \) very close!

\[
E_1 = E_{01} \cos \left[ \frac{2\pi}{\lambda_1} (x - ct) \right] = E_{01} \cos \omega_{\nu_1} \left[ \frac{x}{c} - t \right]
\]

\[
E_2 = E_{01} \cos \left[ \frac{2\pi}{\lambda_2} (x - ct) \right] = E_{01} \cos \omega_{\nu_2} \left[ \frac{x}{c} - t \right]
\]

Same amplitude \( E_{01} \) assumed!

\[
\cos \alpha + \cos \beta = 2 \cos \frac{1}{2} (\alpha + \beta) \cdot \cos \frac{1}{2} (\alpha - \beta)
\]

\[
E_{\text{total}} = 2E_{01} \cos 2\pi \left( \frac{x}{c} - t \right) \left( \frac{\nu_1 + \nu_2}{2} \right) \cdot \cos 2\pi \left( \frac{x}{c} - t \right) \left( \frac{\nu_1 - \nu_2}{2} \right)
\]

\[
\frac{\nu_1 + \nu_2}{2} = \bar{\nu} \quad \text{"average" frequency}
\]

\[
\frac{\nu_1 - \nu_2}{2} = \nu_M \quad \text{"modulation" frequency}
\]

\[
E_{\text{total}} = 2E_{01} \cos 2\pi \nu_M \left[ \frac{x}{c} - t \right] \cdot \cos 2\pi \nu \left[ \frac{x}{c} - t \right]
\]

varies very slowly = "beat" \( \Rightarrow \) "carrier wave"
\[ \bar{\lambda} = \frac{c}{\nu} \]

traveling wave of frequency \( \bar{\nu} = \frac{\nu_1 + \nu_2}{2} \)

having time-varying or modulated amplitude

\[ \lambda_M = \frac{c}{\nu_M} \]

Usually, frequencies \( \nu_1 \) and \( \nu_2 \) vary large (10¹⁴ Hz).

If \( \nu_1 \approx \nu_2 \), then \( \bar{\nu} = \frac{\nu_1 + \nu_2}{2} \Rightarrow \nu_M = \frac{\nu_1 - \nu_2}{2} \)
(modulation frequency much smaller than carrier frequency).

- \( E_1 \)
- \( E_2 \)
- \( E(x) \)
- \( E_p(x) \)
- \( E_{p1}(x) \)
- \( E_{p2}(x) \)
The modulated wave can be treated as a traveling wave of frequency \( v = \frac{v_1 + v_2}{2} \) having a time-varying or modulated amplitude \( E_0(x, t) \), i.e.:

\[
E(x, t) = E_0(x, t) \cos 2\pi \frac{v}{c} \left[ \frac{x}{c} - t \right]
\]

with \( E_0(x, t) = 2 \cdot E_{01} \cos 2\pi \nu_M \left[ \frac{x}{c} - t \right] \).

Intensity of wave is proportional to \( E_0^2(x, t) \):

\[
E_0^2(x, t) = 4 \cdot E_{01}^2 \cos^2 2\pi \nu_M \left[ \frac{x}{c} - t \right]
\]

\[
= 2 \cdot E_{01}^2 \left[ 1 + \cos 2 \cdot 2\pi \nu_M \left( \frac{x}{c} - t \right) \right]
\]

Intensity oscillates about a maximum value \( 2 \cdot E_{01}^2 \) with a frequency \( 2\nu_M = \frac{v_2 - v_1}{c} \) called "beat frequency". Modulation frequency is half the beat frequency!

Beats in light waves first observed in 1955 (Phys. Rev. 99, 1691 (1955)).

Slight shift of frequency of light emission produced by Zeeman effect (sharp spectral line split into 2 components by magnetic field). Both waves recombined at surface of photoelectric mixing tube → beat frequency \( v_2 - v_1 = 10^{10} \text{ Hz} \), corresponding to 3 cm microwaves, generated.
With lasers, light bends easily observed (even low beat frequencies of a few hertz out of 10 kHz) can be used to measure speed of targets (laser radar; doppler effect produces frequency shift of moving target; overlap of outgoing and reflected light → beat frequency).

C) **Group velocity**

purely sinusoidal wave: phase moves with

phase velocity \( v = \frac{c}{n} \).

Question: how fast does modulation envelope move in amplitude modulated carrier wave? how large is "group velocity"? if \( v_1 \) and \( v_2 \) waves travel with equal speed (non-dispersive medium), then group velocity \( \frac{v_g}{v} = \text{phase velocity} \).

In general, however, media are dispersive i.e. \( v_1 \) and \( v_2 \) have different velocities; and the group envelope will move with different velocity as compared to the motion (velocity) of the phase: \( \frac{v_g}{v} = \frac{\frac{\omega}{k}}{\frac{\omega}{k_1} = \frac{\Delta \omega}{\Delta k} = \frac{\omega_{12}}{k_1-k_2} = \frac{\omega}{k} \frac{d\omega}{dk} \)

Since \( \omega = k \nu \Rightarrow v_g = v + k \frac{d\nu}{dk} = \frac{c}{n} - \frac{k \nu}{n^2} \frac{du}{dk} = \frac{c}{n} \left( 1 - \frac{k}{n} \frac{du}{d\lambda} \right) \)}
\( \frac{du}{d\lambda} \) normally < 1 (negative) \( \Rightarrow \) group velocity. Smaller than \( \frac{\lambda}{\eta} \). (In region of anomalous dispersion, when \( \frac{du}{d\lambda} > 0 \), signal propagates with another velocity, "signal velocity", which is smaller than c).
3) Nonperiodic Waves, Pulses and Wave packets

So far, waves were regarded as monochromatic waves (single frequency: $\nu$) with infinite spatial extent and infinite time duration. Such waves never exist in nature. More important: waves of finite spatial and temporal extent:

$\Delta x \text{ or } \Delta t 

These waves are nonperiodic and are best described as "pulses" or wave packets, consisting of a large number of different frequencies.

Very useful in this context: Fourier theorem

Any nonperiodic function $f(x)$ can still be synthesized from an infinite number of monochromatic waves:

$$ f(x) = \frac{1}{\pi} \left[ \int_{0}^{\infty} A(k) \cos kx \, dk + \int_{0}^{\infty} B(k) \sin kx \, dk \right] $$

with $A(k) = \int_{-\infty}^{\infty} f(x) \cos kx \, dx$ and $B(k) = \int_{-\infty}^{\infty} f(x) \sin kx \, dx$

Fourier cosine transform

Fourier sine transform
Example: \( \text{quick} \) \( \cos \text{ine} \) \( \text{\textquotedblleft wave-train\textquotedblright} \)

\[
E(x) = \begin{cases} 
E_0 \cos k_p x & \text{when } -L \leq x \leq L \\
0 & \text{when } |x| > L 
\end{cases}
\]

Fourier analysis:

\( B(k) = 0 \) since \( E(x) \) is even function

\[
A(k) = \int_{-\infty}^{\infty} f(x) \cos kx \, dx \quad = \int_{-L}^{L} E_0 \cos k_p x \cos kx \, dx
\]

\[
= E_0 \cdot \frac{1}{2} \left[ \cos (k_p + k)x + \cos (k_p - k)x \right] \bigg|_{-L}^{L}
\]

\[
= E_0 \cdot L \left[ \frac{\sin (k_p + k)L}{(k_p + k)L} + \frac{\sin (k_p - k)L}{(k_p - k)L} \right]
\]

Since wave-train is not infinitely long, it is composed of a continuous range of spatial frequencies; i.e., principally an infinite number of waves forming a wave-packet or wave-group.
Similar analysis holds for finite wave-train in time domain:

\[ E(t) = \begin{cases} E_0 \cos \omega t & \text{when } -T \leq t \leq T \\ 0 & \text{when } |t| > T \end{cases} \]

The temporal duration of wave-train (pulse) is \( 2T \), and Fourier analysis gives frequency width \( \Delta \omega = \frac{2\pi}{T} \).

For wave-train, the width of the Fourier transform is \( \Delta k = \frac{2\pi}{L} \) or \( \Delta \omega = \frac{2\pi}{L} \). The spatial and temporal extent are \( \Delta x = 2L \) and \( \Delta t = 2T \), respectively.

The product of the width of the wave-packet in \( k \)-space and width in \( x \)-space is \( \Delta k \Delta x = 4\pi \), and analogously \( \Delta \omega \Delta t = 4\pi \).

\( \Delta k \) and \( \Delta \omega \) are termed frequency bandwidths.

Since \( \Delta \omega = \frac{2\pi}{\Delta t} \):

\[ \Delta \nu \sim \frac{1}{\Delta t} \]

If wave-train is short (short pulse), frequency bandwidth is large, and vice versa. Sinusoidal wave of length \( 2L \) (\( \Delta t = \frac{2\pi}{2} \)) is composed of group of waves with frequency bandwidth \( \Delta \nu = \frac{1}{\Delta t} \).
How long is typical wave train?

Consider typical electric dipole allowed emission from excited atom: transition time $\tau = 10^{-8}$ sec. Corresponding wave train has temporal duration $\Delta t = 10^{-8}$ sec, and thus the "natural" line width of the spectral emission is $\Delta \nu = 10^8$ Hz. Spatial extent of this wave train:

$$\Delta x = c \cdot \Delta t = 3 \cdot 10^8 \frac{\text{cm}}{\text{sec}} \cdot 10^{-8} \text{ sec} = 300 \text{ cm}$$

Relations between $\Delta t$, $\Delta x$ and $\Delta \nu$ can also be derived directly from uncertainty principle:

$$\Delta E \cdot \Delta t = h = \Delta p \cdot \Delta x$$
$$h \cdot \Delta \nu \cdot \Delta t = h$$

$$\Rightarrow \Delta x \Delta t = 1 \quad \Delta \left( \frac{1}{\lambda} \right) \Delta x = 1$$

Photon is either certain in space or in momentum, time or in frequency (and not in both simultaneously)!

The time satisfying relation $\Delta t = \frac{1}{\Delta \nu}$ is called \textbf{coherence time}.

The length $\Delta x = c \cdot \Delta t$ is the \textbf{coherence length}.
Examples

a) ultrashort pulse from titanium sapphire laser with 30 femtosecond time duration, emitting laser light near 800 nm in near infrared wavelength range.

\[ \Delta t = 30 \times 10^{-15} \text{ sec} \implies \Delta \nu = 33 \times 10^{12} \text{ Hz} \]

\[ \Delta \lambda = \Delta \left( \frac{c}{\nu} \right) = \frac{\Delta^2}{c} \Delta \nu = \frac{(0.8 \mu \text{m})^2}{3 \times 10^{10} \text{ cm/sec}} \times 33 \times 10^{12} \times \frac{1}{\text{sec}} \]

= 70 nm bandwidth

\[ \Delta x = c \cdot \Delta t = 3 \times 10^{10} \text{ cm/sec} \times 30 \times 10^{-15} \text{ sec} = 9 \times 10^{-4} \text{ cm} \]

= 9 \mu \text{m} coherence length

Uncertainty relation: spatial position well defined energy less well contain

b) white light

frequencies from 0.4 - 0.7 \times 10^{15} \text{ Hz} \implies \Delta \nu = 0.3 \times 10^{15} \text{ Hz}

\[ \Delta t = 3 \times 10^{-15} \text{ sec} \]

\[ \Delta x = 3 \times 10^{-10} \text{ cm/sec} \times 3 \times 10^{-15} \text{ sec} = 9 \times 10^{-5} \text{ cm} = 900 \mu \text{m} \]

White light has coherence length of the order of \( \sim \lambda \)

( Interference experiment is possible with path difference \( \Delta x \approx 1 \)\)
while light can be envisioned as
a) superposition of sinusoidal waves of all frequencies
   over range Δν
b) random succession of short pulses (~ 3 · 10^(-15) sec)
   no frequency uncertainty over whole range of
   visible light.
(available bandwidth of visible light (3 · 10^14 Hz) very
large. TV channel, in comparison, needs 4 MHz
bandwidth. no visible light range could carry 75 million
TV channels!)

Time interval over which light wave resembles sinusoidal
wave: temporal coherence. The longer the coherence
time, the greater the temporal coherence of light source.
Correspondingly, spatial extent over which light wave
oscillates in regular, predictable way: coherence length.
Single point light source: all points on wavefront
are "spatially coherent"; if light source is extended or
more than one point source, waves are in general
not spatially coherent since sources change phase randomly
and rapidly.

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Illustration of temporal and spatial coherence in wavefront propagation pictures:

If wave temporally coherent (Fig. a), all points on successive wavefronts completely correlated (behavior at $P_1$ can be predicted from behavior at $P_4'$). If wave only partially temporally coherent (Fig. b), point source changes frequency from moment to moment no correlation of wave at far away points ($P_1'$, $P_6'$), but some correlation at close-by points ($P_2'$, $P_3'$).

In both figures (a) and (b), points on same wavefront (like $P_1$, $P_2$, $P_3$) completely correlated: spatially coherent. Same spatial coherence if more than one point source with fixed phases and frequencies.

However, if frequencies and phases vary, no spatial or temporal coherence.
E) Conditions for observance of interference

1) Phase difference between interfering waves must be constant over measuring time interval, i.e., waves must be coherent; otherwise, interference pattern exists only for a very short time and so will wash out over required longer observation time.

Principle of interferometric setup: splitting of light from one light source into two (or more) parts, either by division of amplitude or division of wavefronts.

2) Interfering waves must have nearly same frequencies; otherwise, rapidly varying phase difference grows over time, causing $\langle I_{12} \rangle$ to go to zero during detection interval.

3) Clearest interference patterns if interfering waves have same intensities (complete constructive and destructive interference, giving maximum contrast).

4) Two orthogonally polarized waves cannot interfere.
5) After splitting, waves travel through optical pathlengths $S_1 = n_1 L_1$ and $S_2 = n_2 L_2$, and have to be superimposed to observe interference pattern. The difference in optical path lengths $\Delta = L_1 n_1 - L_2 n_2$ determines the phase shift between interfering waves.

If $\Delta = m \lambda$ ($m = 0, 1, \ldots$) $\Rightarrow$ constructive interference;

or $\delta = m \cdot \frac{2\pi}{\lambda}$

If $\Delta = (m + \frac{1}{2}) \lambda$ $\Rightarrow$ destructive interference.

Pathlength difference between waves must be smaller than coherence length to observe interference pattern! ($\Delta < \Delta_c$)