Lecture 3: May 22\textsuperscript{nd} 2009

Physics for Scientists and Engineers II
Gauss’s Law (Karl Friedrich Gauss 1777-1855)

- Example of surface of a cube in a constant electric field resulted in a net flux of zero through that closed surface.
- Today: We will look at a spherical closed surface (shell) concentric around a point charge $q$ and calculate the electric flux through that shell of radius $r$.

\[ \vec{E} \text{ parallel to } d\vec{A} \text{ everywhere on surface } \]
\[ \vec{E} \text{ also has the same magnitude everywhere on the surface of the shell: } E = k_e \frac{q}{r^2} \]

\[ \Rightarrow d\Phi_E = \vec{E}d\vec{A} = EdA \]
\[ \Rightarrow \Phi_E = \int\int_{shell} \vec{E}d\vec{A} = \int_{shell} EdA = E \int_{shell} dA \]
\[ = E4\pi r^2 = k_e \frac{q}{r^2} 4\pi r^2 = k_e 4\pi q \]

\[ \Rightarrow \Phi_E = \frac{1}{4\pi\varepsilon_o} 4\pi q = \frac{q}{\varepsilon_o} \]

$\Phi_E$ is independent of $r$ and proportional to $q$. 
Generalization

For non-spherical closed surfaces surrounding q, the net electric flux is the same as for the spherical concentric shell (just harder to calculate). Example:

\[
\Phi_E = \oint \vec{E} \cdot d\vec{A} = \oint E_d A = \oint E_d A + \oint E_d A
\]

\[
= E_{left} 2\pi r_2^2 + E_{right} 2\pi r_1^2 = k_e \frac{q}{r_2^2} 2\pi r_2^2 + k_e \frac{q}{r_1^2} 2\pi r_1^2 = k_e 4\pi q
\]

\[
\Rightarrow \Phi_E = \frac{1}{4\pi \epsilon_o} 4\pi q = \frac{q}{\epsilon_o}
\]
Generalization

The electric flux through a closed surface surrounding a point charge $q$ equals $\Phi_E = \frac{q}{\varepsilon_o}$

What if multiple charges are inside a surface?

$$\oint \vec{E} \ d\vec{A} = \int \left( \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 + \ldots \right) d\vec{A}$$

(superposition principle of $\vec{E}$)

$$\Rightarrow \oint \vec{E} \ d\vec{A} = \oint \vec{E}_1 \ d\vec{A} + \oint \vec{E}_2 \ d\vec{A} + \oint \vec{E}_3 \ d\vec{A} + \ldots$$

$$= \frac{q_1}{\varepsilon_0} + \frac{q_2}{\varepsilon_0} + \frac{q_3}{\varepsilon_0} + \ldots = \frac{q_{\text{inside}}}{\varepsilon_0}$$
Gauss’s Law

\[ \Phi_E = \oint \vec{E} \, d\vec{A} = \frac{q_{\text{in}}}{\epsilon_o} \]
Example: Electric Field due to an Infinitely Long Uniformly Charged Rod…..doing it the hard way

\[ d \vec{E} = -k_e \frac{\lambda}{r^2} \sin \Theta \hat{i} + k_e \frac{\lambda}{r^2} \cos \Theta \hat{j} \]

\[ = -k_e \frac{\lambda dx}{r^2} \frac{x}{r} \hat{i} + k_e \frac{\lambda dx}{r^2} \frac{a}{r} \hat{j} \]

\[ = -k_e \frac{\lambda x dx}{r^3} \hat{i} + k_e \frac{\lambda a dx}{r^3} \hat{j} \]

\[ = -k_e \frac{\lambda x dx}{(a^2 + x^2)^{3/2}} \hat{i} + k_e \frac{\lambda a dx}{(a^2 + x^2)^{3/2}} \hat{j} \]
Contribution to \( E \) from \( dq \):
\[
d\vec{E} = -k_e \frac{\lambda x \, dx}{(a^2 + x^2)^{3/2}} \mathbf{i} + k_e \frac{\lambda a \, dx}{(a^2 + x^2)^{3/2}} \mathbf{j}.
\]

Total electric field at point \( P \):
\[
\vec{E} = \int_{-\infty}^{+\infty} \left( -k_e \frac{\lambda x \, dx}{(a^2 + x^2)^{3/2}} \mathbf{i} + k_e \frac{\lambda a \, dx}{(a^2 + x^2)^{3/2}} \mathbf{j} \right)
\]
\[
E_x = -k_e \lambda \int_{-\infty}^{+\infty} \frac{x}{(a^2 + x^2)^{3/2}} \, dx \\
E_y = k_e \lambda a \int_{-\infty}^{+\infty} \frac{1}{(a^2 + x^2)^{3/2}} \, dx.
\]
….solving the integrals

\[ E_x = k_e \lambda \int_{-\infty}^{+\infty} \frac{x}{(a^2 + x^2)^{3/2}} \, dx = 0 \quad \quad \quad E_y = k_e \lambda \int_{-\infty}^{+\infty} \frac{a}{(a^2 + x^2)^{3/2}} \, dx \]

Substitution: \( x = a \tan \Theta \) \( \Rightarrow \) \( dx = \frac{a}{\cos^2 \Theta} \, d\Theta \) (I wouldn't expect you to know that)

\[ E_y = k_e \lambda \int_{-90^\circ}^{+90^\circ} \frac{a}{(a^2 + a^2 \tan^2 \Theta)^{3/2}} \frac{a}{\cos^2 \Theta} \, d\Theta = k_e \lambda \int_{-90^\circ}^{+90^\circ} \frac{1}{a} \left( 1 + \tan^2 \Theta \right)^{3/2} \frac{1}{\cos^2 \Theta} \, d\Theta \]

\[ = \frac{k_e \lambda}{a} \int_{-90^\circ}^{+90^\circ} \frac{1}{\cos^2 \Theta} \left( \frac{1}{\cos^2 \Theta} \right)^{3/2} \, d\Theta = \frac{k_e \lambda}{a} \int_{-90^\circ}^{+90^\circ} \cos \Theta \, d\Theta \]

\[ = \frac{k_e \lambda}{a} \left[ \sin \Theta \right]_{-90^\circ}^{+90^\circ} = \frac{k_e \lambda}{a} (\sin(90^\circ) - \sin(-90^\circ)) = \frac{2k_e \lambda}{a} \]
Example: Electric Field due to an Infinitely Long Uniformly Charged Rod.....using Gauss’s Law

By symmetry: \( E_x = 0 \) and \( E_{radial} = \text{const.} \) at any point that is radially a distance of "a" away from the line charge. 
\( \vec{E} \) is perpendicular to the cylindrical round surface everywhere. 
No flux goes through the flat end surfaces of the cylinder.
The charge enclosed in the cylinder is: \( q_{\text{inside}} = \lambda L \)

Gauss's Law applied:

\[
\Phi_E = \oint_{\text{cylinder surface}} \vec{E} \, d\vec{A} = E \oint dA = E(2\pi a L) = \frac{q_{\text{inside}}}{\varepsilon_0} = \frac{\lambda L}{\varepsilon_0}
\]

\[\Rightarrow E(2\pi a L) = \frac{\lambda L}{\varepsilon_0}\]

\[\Rightarrow E = \frac{\lambda L}{2\pi a L \varepsilon_0} = \frac{\lambda}{2\pi a \varepsilon_0} = 2k_e \frac{\lambda}{a}\]

**Question:** Why would Gauss’s law be unpractical to use if we picked a spherical surface in this particular problem?

**Answer:** The electric field vector is not constant along the spherical surface.
Rules for Using Gauss’s Law to Evaluate the Electric Field

- $\vec{E}$ can be argued by symmetry to be constant in magnitude and parallel to $d\vec{A}$ over a part or all of the surface.  \[ \Rightarrow \ E d \vec{A} = E dA \]
- All other surfaces have zero flux going through them ($\vec{E} \cdot d\vec{A} = 0$).
This condition is fulfilled if either $\vec{E} = 0$, or if $\vec{E} \perp d\vec{A}$ on those surfaces.
Example: Spherical Shell with Charge on it. Determine $E(r)$

**Constant volume charge density.**

Total charge $Q$

**Region 1:** $r < a$

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{inside}}}{\varepsilon_0} = 0 \quad \Rightarrow \quad E = 0$$

**Region 2:** $a < r < b$

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{\text{inside}}}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{q_{\text{inside}}}{\varepsilon_0} = k_e \frac{q_{\text{inside}}}{r^2}$$

$$q_{\text{inside}} = \rho V_{\text{inside}} = \rho \left( \frac{4}{3} \pi r^3 - \frac{4}{3} \pi a^3 \right)$$

$$= \frac{Q}{\left( \frac{4}{3} \pi b^3 - \frac{4}{3} \pi a^3 \right)} \left( \frac{4}{3} \pi r^3 - \frac{4}{3} \pi a^3 \right) = Q \frac{r^3 - a^3}{b^3 - a^3}$$

$$\Rightarrow E = k_e \frac{Q}{r^2} \frac{r^3 - a^3}{b^3 - a^3} = k_e \frac{Q}{b^3 - a^3} \left( r - \frac{a^3}{r^2} \right)$$

**Region 3:** $r > b$

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0}$$

$$\Rightarrow E = \frac{Q}{4\pi r^2 \varepsilon_0} = k_e \frac{Q}{r^2}$$
Conductors in Electrostatic Equilibrium

- Electrostatic equilibrium in a conductor: The net motion of charges within the conductor is zero.
- Properties of conductors in electrostatic equilibrium:
  - $E = 0$ inside the conductor (hollow or solid).
  - Charged conductors: Charge is on the surface.
  - Just outside the surface of the conductor:
    \[ \vec{E} \perp \text{surface} \]
    \[ E = \frac{\sigma}{\varepsilon_0}, \text{ where } \sigma \text{ is local surface charge density.} \]
    - Surface charge density is greatest where surface has smallest radius of curvature. (…and therefore $E$ is largest just outside those surface areas.)
E = 0 inside the conductor (hollow or solid).

**How do we know?** Consider an uncharged conductor (or charged, it doesn’t matter)

...and place it into an electric field...

free electrons experience a force (here, to the left side)

...positive charge remains on the right side as negative charge accumulates on the left...

Charges keep separating and creating extra electric field. ....until extra field cancels the external electric field.... ....and the separation of charges stops (electrons on the inside no longer experience a net force.)

$E_{\text{inside}} = 0$
Conductors in Electrostatic Equilibrium

Charged conductors: Charge is on the surface.

**How do we know?** Consider an charged conductor of some shape

We know: \( \vec{E} = 0 \) on the inside

Imagine a Gaussian closed surface just beneath the actual surface.

Since \( \vec{E}_{\text{inside}} = 0 \) \( \Rightarrow \Phi_E = 0 \) through this surface

\( \Rightarrow \) no net charge inside of Gaussian surface = 0

\( \Rightarrow \) entire net charge must reside on the surface
$\vec{E} \perp \text{surface}$

**How do we know?** Suppose $E$ was not perpendicular to surface

Then $E$ would have a component parallel to the surface

$\rightarrow$ Motion of charges would occur along the surface (not yet in electrostatic equilibrium).

$\rightarrow$ In electrostatic equilibrium there cannot be a component of $E$ parallel to the surface.
Conductors in Electrostatic Equilibrium

\[ E = \frac{\sigma}{\varepsilon_0}, \text{ where } \sigma \text{ is the local surface charge density.} \]

**How do we know?** Consider a surface element of area A, small enough to be “flat”

Consider a Gaussian surface (e.g., cylinder with end caps parallel to surface, one end cap on the inside, one just outside).
Calculate flux through the cylinder

\[ \Phi_E = EA = \frac{q_{\text{inside}}}{\varepsilon_0} = \frac{\sigma A}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{\varepsilon_0} \]
Imagine a charge $q_0$ is moved in an electric field by some small displacement:

$$W_{Fe} = F_e \cdot ds = q_0 E \cdot ds$$
Chapter 25: Work Done by Electric Force – Along a Path

- Imagine a charge $q_0$ is moved in an electric field along some path.

Work done by electric force: $W_{Fe} = \int_{\text{path}} \vec{F}_e \cdot d\vec{s} = q_0 \int_{\text{path}} \vec{E} \cdot d\vec{s}$

"path integral" or "line integral"
• In other words: Electrostatic force is a conservative force.

\[ \text{An electric potential (a scalar quantity) can be defined.} \]

\( W_{F_e} = \int_{path} F_e \cdot d\vec{s} = F_e \cdot \Delta \vec{x}_1 + F_e \cdot \Delta \vec{x}_2 + F_e \cdot \Delta \vec{x}_3 + F_e \cdot \Delta \vec{x}_4 + F_e \cdot \Delta \vec{x}_5 = -F_e \cdot a + F_e \cdot a = 0 \)

(Work done by \( \vec{F}_e = 0 \) actually for any closed path)

• In other words: Electrostatic force is a **conservative force**.

\( \Rightarrow \) An **electric potential** (a scalar quantity) can be defined.

(just like we did with the gravitational force / potential)