Physics 7740, Fall 2006

Supplementary material—Bessel Functions

(Frank E. Harris, 15 October 2006)

This document corrects some errors in the text and supplies alternate (and in some cases more elementary) analyses of some of the properties of Bessel functions. We use the notation of the text: the Bessel function of the Second Kind (written Y_ν in most current literature) is here denoted N_ν.

Differential Equation Solutions in Cylindrical Coordinates

Table 9.3 (page 560, Arken & Weber) is misleading and to some extent incorrect. To clarify this, we write

$$\nabla^2\psi + A\psi = 0\,, \tag{1}$$

where A can have either sign, or, for the Laplace equation, zero. Using cyindrical coordinates (ρ, ϕ, z), and taking ψ in the separated form $\psi = P(\rho)\Phi(\phi)Z(z)$, we reach the following three equations:

$$\begin{aligned} \frac{d^2\Phi}{d\phi^2} &= -m^2\Phi\,, \\ \frac{d^2Z}{dz^2} &= Nz\,, \\ \rho^2\frac{d^2P}{d\rho^2} + \rho\frac{dP}{d\rho} + (K\rho^2 - m^2)\rho &= 0\,. \end{aligned}$$

We restrict consideration to situations in which the differential equation is to be solved for the full range of ϕ, $(0 \le \phi < 2\pi)$, and continuity conditions on ϕ restrict Φ to solutions which are periodic in ϕ with period 2π. This restriction causes Φ to be of the form $a_c \cos m\phi + a_s \sin m\phi$ with m an integer, or equivalently, $a_+ \exp(im\phi) + a_- \exp(-im\phi)$. In any case, the ϕ separation constant is $-m^2$, with m real and integral. The Φ solution and the restrictions on m are properly reflected in Table 9.3 and in the text.

We now turn to the z equation. The rationale for using cylindrical coordinates is normally that the differential equation is to be solved for a region with cylindrical boundary conditions, so we consider problems defined by maximum and minimum values of the coordinate z (denoted z_1 and z_2). In that case there will be conditions on Z at these z values, and these conditions determine the solution to be chosen for the z equation. Independently of the value or sign of A (the constant in the differential equation), the z boundary conditions (or the solution to be sought) may cause the separation constant N to be of either sign, meaning that the z solution may either be oscillatory (i.e. $\sin cz$ and/or $\cos cz$, in which case $N = -c^2$) or exponential (i.e. $\exp(cz)$ and/or $\exp(-cz)$, in which case $N = +c^2$). This situation is not properly represented in Table 9.3.

Finally, we examine the ρ equation. The character of its solution is determined by the separation constant K, which, in order to solve the differential equation, must have the value $K = A + N$. Since N has a value (and sign) determined only by the z boundary conditions, the sign of K is not uniquely related to the sign of A, and therefore cannot be assigned as was done in Table 9.3. What can be said is that if K is positive, the ρ equation will have as solutions Bessel functions (J_m or N_m) of argument $k\rho$, where $k = \sqrt{K}$ (and not the argument given in Table 9.3). If K is negative, the solutions to the ρ equation will be the modified Bessel functions (I_m or K_m), with argument $k\rho$, where now $k = \sqrt{-K}$ (again different from Table 9.3). Note in particular that this situation persists even for the Laplace equation (that with $A = 0$). Depending on the boundary conditions, N can have any value (and either sign), and the same is then also true of K. Table 9.3(c) is just wrong, reflecting only the unusual special case $K = 0$.

Considering further the Laplace equation, we make one further observation. If the separation constant N is negative, meaning that $Z(z)$ is oscillatory, we note that K will also be negative and that $P(\rho)$ will then be a **modified** Bessel function (which is inherently non-oscillatory). On the other hand, a positive value of N will cause $Z(z)$ to be non-oscillatory, with $P(\rho)$ an ordinary Bessel function, with oscillatory behavior. Since oscillatory functions have curvature that brings the function value back toward zero while non-oscillatory functions do not, it can be seen that the different behaviors of solutions to the Laplace equation in the z and ρ coordinates are consistent with the well-known fact that the solutions cannot exhibit a maximum. In electrostatics this corresponds to the observation that the potential cannot have a maximum in the interior of a charge-free region.

Series for Neumann Functions

The discussion in Section 11.3, particularly that leading to Eqs. (11.62) and (11.64), is incomplete and to some extent misleading. We consider here the series expansion of N_0, obtained as the $\nu \to 0$ limit of N_ν. From the definition

$$N_\nu(x) = \frac{\cos \nu\pi \, J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}\,, \tag{2}$$

and the series for J_ν,

$$J_\nu(x) = \sum_{s=0}^{\infty} \frac{(-1)^s}{s!\,\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{\nu+2s}, \tag{3}$$

we take the limit by applying l'Hôpital's rule, i.e. taking derivatives of the numerator and denominator of the expression for N_ν. Keeping the dependence on ν explicit (to make the analysis more transparent), we have

$$\begin{aligned} N_0(x) = {} & \frac{1}{\pi\cos\nu\pi} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left[\frac{\cos\nu\pi}{\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{\nu+2s} \ln\left(\frac{x}{2}\right) - \frac{\pi\sin\nu\pi}{\Gamma(\nu+s+1)} \left(\frac{x}{2}\right)^{\nu+2s} \right. \\ & \left. + \cos\nu\pi \left(\frac{x}{2}\right)^{\nu+2s} \frac{\partial}{\partial\nu}\left(\frac{1}{\Gamma(\nu+s+1)}\right) \right] \\ & - \frac{1}{\pi\cos\nu\pi} \sum_{s=0}^{\infty} \frac{(-1)^s}{s!} \left[-\frac{1}{\Gamma(-\nu+s+1)} \left(\frac{x}{2}\right)^{-\nu+2s} \ln\left(\frac{x}{2}\right) \right. \\ & \left. + \left(\frac{x}{2}\right)^{-\nu+2s} \frac{\partial}{\partial\nu}\left(\frac{1}{\Gamma(-\nu+s+1)}\right) \right]. \end{aligned} \tag{4}$$

The derivative involving the gamma function reduces as illustrated here:

$$\begin{aligned} \frac{\partial}{\partial\nu}\left(\frac{1}{\Gamma(\nu+s+1)}\right) &= -\frac{1}{\Gamma(\nu+s+1)} \frac{d\ln\Gamma(\nu+s+1)}{d\nu} \\ &= -\frac{1}{\Gamma(\nu+s+1)} \left[-\gamma + \sum_{n=1}^{\infty} \frac{\nu+s}{n(\nu+s+n)} \right]. \end{aligned} \tag{5}$$

Here γ is the Euler-Mascheroni constant. For integer ν, the summation over n reduces to $1 + \frac{1}{2} + \ldots + \frac{1}{\nu+s}$ (or zero if $\nu + s = 0$).

Setting now $\nu = 0$, Eq. (4) simplifies greatly, in part because the terms involving $\ln(x/2)$ can be recognized as containing the summation defining J_0. The final result is

$$N_0(x) = \frac{2}{\pi}\left(\ln\left(\frac{x}{2}\right) + \gamma\right) J_0(x) - \frac{2}{\pi} \sum_{s=1}^{\infty} \frac{(-1)^s}{s!s!} \left(\frac{x}{2}\right)^{2s} \left(1 + \frac{1}{2} + \ldots + \frac{1}{s}\right). \tag{6}$$

The leading terms of this expansion are

$$N_0(x) = \frac{2}{\pi}\left(\ln\left(\frac{x}{2}\right) + \gamma\right) J_0(x) + \frac{2}{\pi}\left(\frac{x}{2}\right)^2 - \frac{2}{\pi}\frac{1}{2!2!}\left(\frac{x}{2}\right)^4 \left(\frac{3}{2}\right) + \ldots. \tag{7}$$

This expression contains terms that are larger than the $O(x^2)$ remainder used to cut off the expansion in Eq. (11.62).

Generalization to nonzero integers n yields

$$N_n(x) = \frac{2}{\pi}J_n(x)\left(\ln\left(\frac{x}{2}\right)+\gamma\right) - \frac{1}{\pi}\sum_{s=0}^{n-1}\frac{(n-s-1)!}{s!}\left(\frac{x}{2}\right)^{2s-n}$$
$$-\frac{1}{\pi}\sum_{s=0}^{\infty}\frac{[H_s+H_{n+s}]}{s!(n+s)!}(-1)^s\left(\frac{x}{2}\right)^{n+2s}. \quad (8)$$

Here $H_k = \sum_{j=1}^{k} j^{-1}$, with $H_0 = 0$. The H_k are sometimes called *harmonic numbers* as they are initial terms of the harmonic series. The formula given as Eq. (8) differs significantly from Eq. (11.64).

Asymptotic Behavior of Bessel Functions

The asymptotic behavior to be examined is that of $Z_\nu(x)$ as $x \to \infty$ at fixed ν; the formulas become more accurate for large x/ν. Here Z_ν is any ordinary or modified Bessel function (J_ν, N_ν, $H_\nu^{(1)}$, $H_\nu^{(2)}$, I_ν, K_ν). The standard way to develop these asymptotic formulas involves integral representations containing contour integrals, and to start with the Hankel functions. The analysis in Chapter 11 of the text is flawed because it assumes we already know that K_ν asymptotically approaches zero at large x (this fact is used to determine that a particular integral representation satisfying the modified Bessel equation describes K_ν, and does not describe either I_ν or a linear combination of I_ν and K_ν). We offer here an alternate procedure that does not involve the use of an integral representation.

We start our analysis by examining $K_\nu(x)$, for x large and real. To do so, we need to know the following:

1. $K_\nu(x)$ is a positive, monotone increasing function of ν for positive ν (at fixed x).
2. The asymptotic values of $K_\nu(x)$ become independent of ν in the large-x limit.
3. That the asymptotic value of $K_\nu(x)$ can be determined directly for one value of ν (this value is $\nu = 1/2$).

We will not prove Item 1 here; its validity can be confirmed "experimentally" by examining tables of K_ν. To establish Item 2, look at the K_ν recurrence formula, in the form

$$K_{\nu+1}(x) = K_{\nu-1}(x) + \frac{2\nu}{x}K_\nu(x). \quad (9)$$

Since K_ν is smaller than $K_{\nu+1}$, in the limit $\nu/x \to 0$, the ratio of $K_{\nu+1}$ to $K_{\nu-1}$ will approach unity, and because of the monotonicity, the same must be true for all K of orders between $\nu - 1$ and $\nu + 1$ and by extension, to any ν, providing x is sufficiently large. In other words, the asymptotic value of K_ν is independent of ν.

We now ask for the specific value of $K_{1/2}(x)$. The series expansion for $I_{1/2}$ is

$$I_{1/2}(x) = \sum_{s=0}^{\infty} \frac{1}{s!\,\Gamma(s+\frac{3}{2})} \left(\frac{x}{2}\right)^{2s+\frac{1}{2}} . \tag{10}$$

This can be simplified by noting

$$\Gamma\left(s+\frac{3}{2}\right) = \Gamma(1/2)\left(\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2s+1}{2}\right) = \frac{\sqrt{\pi}(2s+1)!!}{2^{s+1}} = \frac{\sqrt{\pi}(2s+1)!}{2^{2s+1}s!} . \tag{11}$$

Substituting into Eq. (10),

$$I_{1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sum_{s=0}^{\infty} \frac{x^{2s+1}}{(2s+1)!} = \left(\frac{2}{\pi x}\right)^{1/2} \sinh x . \tag{12}$$

A similar analysis can be made for $I_{-1/2}$, leading to

$$I_{-1/2}(x) = \left(\frac{2}{\pi x}\right)^{1/2} \sum_{s=0}^{\infty} \frac{x^{2s}}{(2s)!} = \left(\frac{2}{\pi x}\right)^{1/2} \cosh x . \tag{13}$$

Inserting these expressions into the formula for $K_{1/2}$,

$$K_{1/2}(x) = \frac{\pi}{2}\left(\frac{2}{\pi x}\right)^{1/2} \frac{\cosh x - \sinh x}{\sin \pi/2} = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} . \tag{14}$$

Remembering that we have shown the asymptotic form to be independent of ν, we have

$$K_\nu(x) \sim \left(\frac{\pi}{2x}\right)^{1/2} e^{-x} . \tag{15}$$

Asymptotic forms for the other Bessel functions can now be obtained as in the text. Keeping only the leading terms, and combining powers of i into complex exponentials using the relation $i = \exp(i\pi/2)$, we find

$$\begin{aligned}
H_\nu^{(1)}(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \exp\left(i\left[x - \left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]\right) , \\
H_\nu^{(2)}(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \exp\left(-i\left[x - \left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right]\right) , \\
J_\nu(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \cos\left[x - \left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right] , \\
N_\nu(x) &\sim \left(\frac{2}{\pi x}\right)^{1/2} \sin\left[x - \left(\nu+\frac{1}{2}\right)\frac{\pi}{2}\right] , \\
I_\nu(x) &\sim (2\pi x)^{-1/2} e^{x} .
\end{aligned}$$

Spherical Wave Expansion

An important application of spherical Bessel functions is the so-called spherical wave expansion. This expansion expresses a function of the form $\exp(i\mathbf{r}\cdot\mathbf{p})$, which can be identified as the spatial dependence of a plane wave, in terms of spherical Bessel functions. The expansion takes the form

$$e^{i(\mathbf{r}\cdot\mathbf{p})} = \sum_{n=0}^{\infty} i^n (2n+1) j_n(rp) P_n(\cos\theta)\,, \tag{16}$$

where r and p are the magnitudes of $\mathbf{r}$ and $\mathbf{p}$, and θ is the angle between $\mathbf{r}$ and $\mathbf{p}$.

The spherical wave expansion follows directly from the following integral representation of j_n (for non-negative integer n):

$$j_n(x) = \frac{i^{-n}}{2}\int_{-1}^{1} e^{ixt} P_n(t)\,dt\,, \tag{17}$$

where P_n is a Legendre polynomial. Equation (17) can be proved by verifying that it is correct for j_0 and j_1, for which the integrations are trivial, and then showing that this formulation of j_n satisfies the recurrence relation, (11.161) of the text. We do so by forming $j_{n-1}+j_{n+1}$, integrating by parts, and then using the Legendre function identity (12.23) of the text:

$$\begin{aligned}
j_{n-1}(x)+j_{n+1}(x) &= \frac{i^{-n+1}}{2}\int_{-1}^{1} e^{ixt}\left[P_{n-1}(t)-P_{n+1}(t)\right]\,dt \\
&= \frac{i^{-n}}{2x}\left\{\left[e^{ixt}\left[P_{n-1}(t)-P_{n+1}(t)\right]\right]_{t=-1}^{t=+1} - \int_{-1}^{1} e^{ixt}\left[P'_{n-1}(x)-P'_{n+1}(x)\right]\,dt\right\} \\
&= \frac{i^{-n}}{2x}\left\{0+\int_{-1}^{1} e^{ixt}(2n+1)P_n(t)\,dt\right\} \;=\; \frac{2n+1}{x} j_n(x)\,.
\end{aligned} \tag{18}$$

To obtain the spherical wave expansion, we simply expand $\exp(i\mathbf{r}\cdot\mathbf{p}) = \exp(irp\cos\theta)$ as a series of Legendre polynomials:

$$e^{irp\cos\theta} = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)\,, \tag{19}$$

and note that (writing $\cos\theta = t$)

$$c_n = \frac{2n+1}{2}\int_{-1}^{1} e^{irpt} P_n(t)\,dt\,. \tag{20}$$

Using Eq. (17), the proof of Eq. (16) is immediate.