Tutorial—Expansions in orthogonal functions

For the discussion here, it is convenient to introduce a compact notation for the integrals involved. Let

$$\langle f|\, g\rangle \equiv \int_a^b f(x)g(x)w(x)\,dx\,,$$

where a and b are the interval over which a set of functions is orthogonal, and $w(x)$ is the weight factor involved.

Let $\phi_n(x)$ be a set of functions which is orthogonal, so that

$$\langle \phi_n|\, \phi_m\rangle = 0, \qquad n \neq m.$$

We wish to obtain the expansion of a function $f(x)$ in the orthogonal functions ϕ_n:

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x)\,.$$

The orthogonality makes it easy to determine the coefficients a_n:

$$\langle \phi_m|\, f\rangle = \sum_{n=0}^{\infty} a_n \langle \phi_m|\, \phi_n\rangle = a_m \langle \phi_m|\, \phi_m\rangle\,,$$

because all the terms of the summation with $n \neq m$ vanish because of the orthogonality of the ϕ_n. We solve for a_m:

$$a_m = \frac{\langle \phi_m|\, f\rangle}{\langle \phi_m|\, \phi_m\rangle}\,.$$

If the ϕ_n are normalized as well as orthogonal (an **orthonormal** set), the denominator in the above equation can be replaced by unity.

An important feature of the above is that the coefficients can be determined independently, and their values do not depend upon the number of terms it is planned to use in the expansion.

See also the tutorial on Gram-Schmidt orthonormalization and the Maple code for working with orthonormal expansions.