Problem 1  Relativistic Kinematics

Anna and Bob are on the Starship Enterprise doing an experiment where they bounce a beam of electrons back and forth. They stand at either end of a 10 m hall aligned along the Enterprise’s direction of motion, with Anna towards the rear of the ship and Bob towards its front. The following sequence of events occurs:

L: Anna launches the electron beam toward Bob at time 0.

M: Bob receives the beam at time 20 m/c and bounces it back toward Anna.

N: Anna receives the electron beam from Bob at time 40 m/c.

At the exact time of event L, a Klingon ship passes Anna moving in the same direction as the Enterprise (everyone knows that Klingon Warbirds are faster than Federation ships!).

(a) (5 points) What is the proper time interval between events L and M? Leave your answer in terms of m/c.

The proper time interval, \( \tau_0 \), is defined to be the time interval between two events that happen at the same position in space. In the Enterprise frame, events L and M clearly don’t happen at the same place, so \( \Delta t_{LM} = 20 \text{ m/c} \), is NOT the proper time. The simplest way to calculate \( \tau_0 \) is to use the invariant interval, which is defined as \( (\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2 \), where \( \Delta t = 20 \text{ m/c} \) is the time interval between events L and M in the Enterprise frame, and \( \Delta x = 10 \text{ m} \) is the spatial interval between L and M in the Enterprise frame. Since \( (\Delta s)^2 \) is invariant, it must be the same in any frame. In particular, it must be the same in a frame in which L and M happen at the same location, that is where \( \Delta x' = 0 \). Putting this all together, we have:

\[
(\Delta s)^2 = (\Delta s')^2 \implies (c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2
\]

\[
20^2 - 10^2 = (c\Delta t')^2 - 0, \tag{1}
\]

where the right hand side of the equation is for a frame in which the L and M happen at the same place. In this particular frame, the time interval is precisely the proper time interval, so \( \Delta t' = \tau_0 \). Working out the math, we find \( \tau_0 = \sqrt{300} = 17.3 \text{ m/c} \). Note that \( \tau_0 < \Delta t \) as it must: the proper time interval is the shortest time interval possible between two events that are causally related.

Another way to think about this is to realize that the two events L and M happen at the same spatial location in the electron’s rest frame, and thus the time interval between L and M in this frame is the proper time interval: \( \Delta t_{\text{electron}} = \tau_0 \). If we then regard events L and M as two ticks on the electron’s internal ‘clock’, we can use the normal
time dilation formalism. The critical thing here is to realize that the time interval in the electron’s frame will be shorter than the time interval in the lab frame according to the rule of thumb, “moving clocks go slow.” In this rule, we must recognize that the word “go” means “progress” or “run”, so less time ticks off the electron’s clock compared to the lab’s clock. With this understanding, we can use \( \Delta t_{LM} = \gamma \tau_0 \rightarrow \tau_0 = \Delta t_{LM} / \gamma \), where \( \gamma = 1 / \sqrt{1 - 0.5^2} \) is the Lorentz factor associated with the electron’s motion in the lab frame. This gives the same answer as above, as it must!

(b) (5 points) In the Klingon ship’s frame, events \( L \) and \( M \) are separated by this proper time interval. Given this, what is the speed of the Klingon ship with respect to the Enterprise? Leave your answer in terms of \( c \).

The question is revealing that the two events \( L \) and \( M \) must happen at the same location in the Klingon ship’s frame (otherwise, they wouldn’t measure the proper time). Logically, the only way for this to happen is for the Klingon ship to move at the same speed as the electron beam. In moving from Anna to Bob in the lab frame, the electron beam travels a distance \( \Delta x = 10 \text{ m} \) in a time interval \( \Delta t = 20 \text{ m}/c \). Thus, the electron beam travels with speed \( u_e = \Delta x / \Delta t = 0.5c \), and so does the Klingon ship!

This question can also be answered formally by doing a Lorentz transformation (LT) from the lab frame to the Klingon ship’s (or electron’s) rest frame. In particular, we can use the LT equation for \( \Delta x' \): \( \Delta x' = -\gamma \beta (c \Delta t) + \gamma (\Delta x) \). But this must be zero (events \( L \) and \( M \) happen at the same place in the Klingon frame). So we have \( \gamma \beta (c \Delta t) = \gamma (\Delta x) \), and again we find that \( \beta = \Delta x / (c \Delta t) = 0.5 \).

(c) (5 points) What is the distance between Anna and Bob in the Klingon ship’s frame?

Since Anna and Bob are moving with respect to the Klingon frame, the distance between them will be contracted: \( L = L_0 / \gamma \), where \( L_0 = 10 \text{ m} \) is the proper distance between Anna and Bob, which is measured in the Enterprise frame (where Anna and Bob are at rest). In part (b) above, we found that \( \beta = 0.5 \), so \( \gamma = \sqrt{1 - 0.5^2} = \sqrt{3}/2 \). Thus, in the Klingon frame, the distance between Anna and Bob is \( 5\sqrt{3} = 8.66 \text{ m} \).

(d) (5 points) What are the time and space coordinates of event \( N \) in the Klingon ship’s frame? (Hint: A Lorentz transformation from the Enterprise to the Klingon Ship’s frame is useful here...)

In the Klingon frame, we can (arbitrarily) set the coordinates for event \( L \) to be \((ct', x') = (0, 0)\); that is, the clock on board the Klingon ship is synchronized to the Enterprise clock at \( t = t' = 0 \) when the Klingon ship passes the Enterprise. With this assignment, we can then do an LT on the coordinates of \( N \) from the Enterprise frame \((N : (ct, x) = (40, 0) \text{ m})\) to the Klingon frame. Using the ‘standard’ LT matrix, we then have: \( ct' = \gamma (ct) - \gamma \beta (ct) \) and \( x' = -\gamma \beta (ct) + \gamma (x) \). Using \( \beta = 1/2 \) and \( \gamma = 2/\sqrt{3} \), we find \( ct' = 2(ct) / \sqrt{3} - x / \sqrt{3} = 46.19 \text{ m} \), and \( x' = -(ct) / \sqrt{3} + 2x / \sqrt{3} = 40 / \sqrt{3} = -23.09 \text{ m} \). So in the Klingon frame, the coordinates for \( N \) are: \((ct', x') = (46.19, -23.09) \text{ m}\).

Note that the time interval between events \( L \) and \( N \) is longer in the Klingon ship’s frame (46.19 m/c) than in the laboratory frame (40 m/c). This makes sense, since events \( L \) and \( N \) happen at the same location in the lab frame (at Anna) and thus the time interval between them in the lab frame constitutes the proper time interval. In the Klingon ship’s frame, the two events do not happen at the same location, and thus the time interval between them must
be longer! Compare this with the time interval between events $L$ and $M$ discussed above in part (a): in that case, the electron has the proper time interval and the time interval in the lab frame is longer.

(e) (5 points) You now have everything you need to draw a space-time diagram showing events $L$, $M$, and $N$ in the Klingon ship’s frame. Draw the diagram to scale on the grid below: indicate the slopes of all worldlines and the coordinates of all events.


**Problem 2  Inverse Compton Scattering**

In the laboratory frame, a very low-energy photon of wavelength $\lambda$ collides head-on with a relativistic electron of speed $u_e/c = \beta_e$, as shown in the figure. The collision reverses the propagation direction of the photon and increases its energy, resulting in a final photon wavelength $\lambda'$ and final electron speed $\beta_e'$. **Follow the steps below to find the final photon wavelength, $\lambda'$, in terms of the initial photon wavelength, $\lambda$, and the initial electron speed, $\beta_e$ (and/or $\gamma_e$).**

(a) **(3 points)** Write down the initial 4-momentum of the photon in the laboratory frame.

The energy of a photon is given by $E = hc/\lambda$, and its momentum is $p = E/c = h/\lambda$. Since the photon initially travels in the $-x$ direction, we have $p_x = -h/\lambda$, and $p_y = p_z = 0$. So the initial 4-momentum of the photon in the lab frame is given by:

$$\vec{P}_{i, \text{lab}} = (E/c, p_x, p_y, p_z) = \left( h/\lambda, -h/\lambda, 0, 0 \right). \tag{3}$$

(b) **(6 points)** Use a Lorentz transformation to find the initial 4-momentum of the photon in the rest frame of the electron. Write your answer in terms of $\gamma_e$, $\beta_e$, and $\lambda$.

The electron moves to the right in the lab frame with velocity $\beta_e$, giving $\gamma_e = 1/\sqrt{1 - \beta_e^2}$. To transform the initial 4-momentum to the electron’s rest frame, we must use the Lorentz transformation (LT) matrix with $\beta = \beta_e$ and $\gamma = \gamma_e$. So,

$$\vec{P}_{i, \text{electron}} = \gamma_e \begin{pmatrix} \gamma_e & -\beta_e \gamma_e & 0 & 0 \\ -\beta_e \gamma_e & \gamma_e & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} h/\lambda \\ -h/\lambda \\ 0 \\ 0 \end{pmatrix} \tag{4}$$

$$= \gamma_e (1 + \beta_e) \frac{h}{\lambda} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \tag{5}$$

The subscript $i, \text{electron}$ signifies the initial 4-momentum of the photon in the electron’s rest frame.

(c) **(3 points)** Now we have a collision of a photon and an at-rest electron, and the direction of the photon is reversed (as specified by the problem). We will assume that the photon wavelength in the electron’s rest frame does not change due to this collision. (This is valid when the initial photon wavelength is very long, so that the small Compton shift can be neglected.) Using this information, write down the final 4-momentum of the photon in the electron’s rest frame.
Under the assumptions described, the collision simply reverses the direction of the photon without changing its energy or wavelength, so we just have to change the sign of $p_x$ in the 4-momentum above:

$$\vec{P}_{f, \text{electron}} = \gamma_e(1 + \beta_e) \frac{h}{\lambda} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

The subscript $f, \text{electron}$ signifies the final 4-momentum of the photon in the electron’s rest frame.

(d) (6 points) Use a Lorentz transformation to find the final 4-momentum of the photon in the laboratory frame. Write your answer in terms of $\gamma_e$, $\beta_e$, and $\lambda$.

To go back to the lab frame, we use the same transformation factors, $\beta_e$ and $\gamma_e$, but we must reverse the sign of $\beta_e$ since the laboratory frame moves to the left with respect to the electron. So we have:

$$\vec{P}_{f, \text{lab}} = \gamma_e(1 + \beta_e) \frac{h}{\lambda}$$

$$= \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \gamma_e^2(1 + \beta_e)^2 \frac{h}{\lambda}$$

(e) (3 points) Extract the final wavelength of the photon $\lambda'$ in the laboratory frame in terms of $\gamma_e$, $\beta_e$, and $\lambda$.

The final momentum of the photon in the lab frame is $h/\lambda'$. Equating this with the momentum component (the 2nd component) of the final 4-momentum above gives:

$$\frac{h}{\lambda'} = \gamma_e^2(1 + \beta_e)^2 \frac{h}{\lambda}$$

$$\Rightarrow \lambda' = \frac{\lambda}{\gamma_e^2(1 + \beta_e)^2}.$$ 

(f) (4 points) The cosmic microwave background (CMB) radiation is composed of very low energy photons of wavelength $\sim 2$ mm (yes, millimeters) left over from the big bang. These photons occasionally collide with highly relativistic electrons that have been accelerated by supernovae and other cosmic events. Assuming these electrons have kinetic energy $K_e = 4.60$ MeV, find the wavelength of the CMB photons after the collision.

We simply have to apply the formula we found in part (e) above. First, we must find $\gamma_e$ using the information given. In particular, we have $K_e = (\gamma_e - 1)m_ec^2$ so $\gamma_e = K_e/(m_ec^2) + 1$. Using $m_ec^2 = 0.511$ MeV gives $\gamma_e = 10$. We see that the
electrons are highly relativistic, so we can approximate $\beta_e \simeq 1$ (the actual value is $\beta_e = \sqrt{99}/10 = 0.995$). Now we can apply the formula:

$$\lambda' = \frac{\lambda}{\gamma_e^2(1 + \beta_e)^2}$$

$$= \frac{2 \text{ mm}}{10^2(1 + 1)^2}$$

$$= \frac{2 \text{ mm}}{400}$$

$$= 5 \mu\text{m}.$$