Problem 1  

(10 pts.) Solar fusion

The net result of the fusion reaction that fuels the Sun is to turn four protons and two electrons into one helium nucleus, through the reaction

\[ 4p^+ + 2e^- \rightarrow ^4He^{++} + \text{photons and neutrinos}. \]  

Neutrinos and photons are mass-less particles, and you can think of them as missing energy showing up in the reaction.

The masses of the relevant nuclei are \( m_p = 938.3 \text{ MeV}/c^2 \), \( m_e = 0.511 \text{ MeV}/c^2 \) and \( m_{^4He} = 3725.8 \text{ MeV}/c^2 \). How much energy is released in photons and neutrinos when a kilogram of protons combines with just enough electrons to fuse completely and form helium?

Problem 2  

(10 pts.) Electron-positron system

An anti-electron, also known as a positron, of mass \( m_e \) and speed \( v_1 = 0.6c \), collides with an electron (also of mass \( m_e \)) whose speed is \( v_2 = -0.4c \), to form a bound state called positronium.

(a) Assuming that the positronium has mass \( M \), show that the conservation of energy and momentum requires \( M > 2m_e \).

(b) Compute \( M \) and the speed \( V \) of the positronium state.

(c) Consider the frame in which the electron is initially at rest. Show that the total momentum is conserved also in this frame, and that the mass \( M \) is the same as the one obtained in part (b).
Problem 3  (10 pts.) Nuclear recoil

A stationary excited nucleus of mass $m_e$ decays to its ground state, of mass $m_g$, by emitting a gamma-ray photon of energy $E_\gamma$. The ground state nucleus recoils in the opposite direction at speed $v$. Show that when $v \ll c$ the change of mass of the nucleus is approximately

$$m_e - m_g \approx \frac{E_\gamma}{c^2} \left(1 + \frac{v}{2c}\right). \tag{2}$$

Problem 4  (10 pts.) Pion decay

The neutral pion, or $\pi^0$ meson, is a particle of mass $m_{\pi} = 135 \text{ MeV}/c^2$ that decays into two photons through the reaction $\pi^0 \rightarrow 2\gamma$.

(a) What is the energy of a single photon after the decay, in the rest frame of the pion? What angle do the trajectories of the two emitted photons make with each other?

(b) The reaction is observed in a laboratory, where pions travel at speed $v = 0.995c$. Assuming that the resulting photons move in opposite directions along the $\pi^0$ mesons original line of motion, find the two different energies of the photons $E_1, E_2$ in the laboratory.

(c) In part (b), use the invariance of the energy-momentum vector to show that the energies of the two photons satisfy $m_{\pi} c^2 = 2\sqrt{E_1 E_2}$. Hint: compute the invariant in pion rest frame, then equate it to the same invariant computed for the two photons in the laboratory frame.

Problem 5  (10 pts.) $\Lambda^0$ production

The $K^-$ meson and the $\Lambda^0$ hyperon are two commonly encountered unstable particles. The reaction

$$K^- + p^+ \rightarrow \Lambda^0 + \pi^0, \tag{3}$$

can be used to make $\Lambda^0$ at rest in the laboratory by scattering $K^-$ mesons off a stationary proton target. In the following use $m_p c^2 = 939 \text{ MeV}$, $m_K c^2 = 494 \text{ MeV}$, $m_{\pi} c^2 = 135 \text{ MeV}$ and $m_{\Lambda} c^2 = 1116 \text{ MeV}$.

(a) Find the energy of the incident $K^-$ beam required to just produce $\Lambda^0$ hyperons at rest in the lab.

(b) What is the $\pi^0$ energy corresponding to this reaction?