From the relativistic expression for the energy

$$E^2 = \vec{p}^{\,2}c^2 + m^2c^4,$$

we see that the quantity

$$\frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2,$$

is invariant under Lorentz transformations. A different way of expressing this statement is to say that energy and momentum make up a four-vector

$$p^\mu = \left(\frac{E}{c}, p_x, p_y, p_z\right).$$

In general, any quantity that under Lorentz transformations behaves like the components of the vector (ct, x, y, z) is called a four vector. Since the energy and the momentum form a four vector the length

$$p_\mu p^\mu = \frac{E^2}{c^2} - \vec{p}^{\,2} = m^2c^2,$$

is invariant under Lorentz transformations. This is, of course, the case since m is the rest mass and c does not change under Lorentz transformations.

The reason for introducing the energy-momentum four vector is that this quantity is conserved in any physical process in which no external force, such as gravity or friction, has an influence. This will always be the case if we are analyzing a collision between elementary particles which occurs so fast that any external force can be neglected. So imagine for example that we have a collision in which two particles, which we denote by A and B collide and as a result two other particles result, which we call C and D:

$$A + B \rightarrow C + B.$$

The conservation of the relativistic energy and momentum four vector then says

$$p_A^\mu + p_B^\mu = p_C^\mu + p_D^\mu.$$

In components this implies that the energy is conserved

$$E_A + E_B = E_C + E_D,$$

and the momentum is conserved

$$\vec{p}_A + \vec{p}_B = \vec{p}_C + \vec{p}_D.$$

Note that the energy in a relativistic collision is conserved but the rest mass will not be conserved. In order to see this consider the following examples takes out of the book by D. Griffiths

Example 1: Two clumps of clay, each of mass m, collide head on at a velocity $3c/5$. They stick together. The question is to compute the mass M of the final composite lump.

We apply the conservation of the energy and momentum four vector to get

$$E_1 + E_2 = E_M = Mc^2 \qquad \text{and} \qquad \vec{p}_1 + \vec{p}_2 = \vec{p}_M = 0,$$

where the last equality holds since the final lump is at rest. The conservation of energy then implies

$$Mc^2 = E_1 + E_2 = \frac{2mc^2}{\sqrt{1-(3/5)^2}} = \frac{5}{4}(2mc^2).$$

This result makes it clear that the rest mass is no longer conserved in process governed by special relativity. The mass of the final lump is bigger than the sum of the masses of the two initial lumps. The kinetic energy of the initial lumps was transformed to mass.

This example shows that we can interpret mass as a concentrated form of energy. If one converts mass into energy using

$$E = mc^2,$$

then one sees that a small amount of mass generates huge amounts of energy. So for example if we could convert a mass of $200lb$ completely into energy we would generate 2×10^{15}Wh which would correspond to 10% of the entire US energy consumption per year. But in many situation only a very small percentage of the rest mass will be released in the form of energy.

Example 2: A particle with mass M at rest decays into two pieces, each of mass m. What is the speed of each piece as it flies off?

We again apply the conservation of the energy and momentum four vector. The conservation of momentum says that the two lumps fly off in opposite directions at equal speeds. The conservation of energy requires

$$M = \frac{2m}{\sqrt{1 - \frac{v^2}{c^2}}},$$

which implies

$$v = c\sqrt{1 - \left(\frac{2m}{M}\right)^2}.$$

This result only makes sense if the original mass M is larger than $2m$. If M is smaller than $2m$ there is not enough energy available to generate the rest mass of the two resulting particles. One says that $M = 2m$ is the threshold for the process $M \to 2m$ to occur. The deuteron is below the threshold for decay into a proton and a neutron since the corresponding masses are

$$m_d = 1875.6MeV/c^2 \qquad \text{and} \qquad m_p + m_n = 1877.9MeV/c^2,$$

which means that it is stable.

On the other hand, a pion π (with mass $m_\pi = 139.6\ MeV/c^2$) in its rest frame, can decay into a muon ($m_\mu = 105.7\ MeV/c^2$) and an almost massless neutrino

$$\pi^+ \to \mu^+ + \nu_\mu.$$

The pion disappears releasing its rest energy. Part of this energy will be used to generate the lighter muon and the the other part will be transformed into kinetic energy of the muon and the neutrino. This makes it clear that when a particle decays the resulting particles must have a smaller total mass.

In the big accelerators that we discussed last week, elementary particles are accelerated to very high velocities. When the particle beams collide the high kinetic energy is then used to generate particles with very high mass. This is an example in which kinetic energy is transformed to mass.

Another consequence of the relativistic relation between the energy and the momentum is the following. We had derived the very important relation between the energy and the momentum in special relativity, namely

$$E^2 = \vec{p}^{\,2} c^2 + m^2 c^4.$$

Taking the square root

$$E = \pm c \sqrt{\vec{p}^{\,2} + m^2 c^2},$$

we see that two signs are allowed. In classical physics we can just ignore the states with a negative energy. A particle with a negative kinetic energy just does not exist. But, as we will later see, in a relativistic quantum field theory the negative energy states have to be taken seriously and they correspond to antiparticles. The prediction that there is a doubling in the world of elementary particles, i.e. that to every particle there exists an antiparticle and vice-versa, is one of the big triumphs or relativistic quantum field theory.

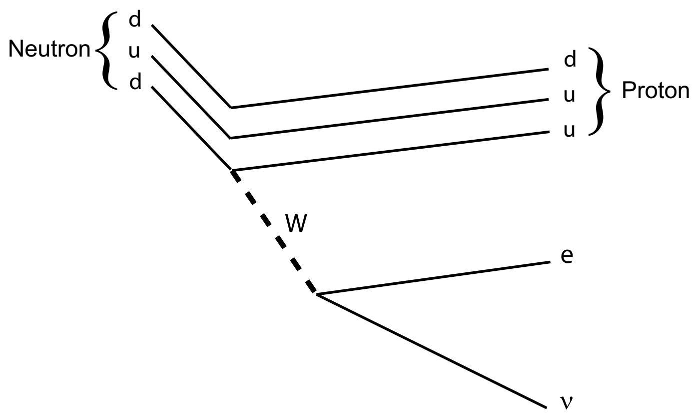
2.3 The Subatomic Particles and their Interactions

Until now we have discussed spin and mass as properties that characterize elementary particles. Nowadays many elementary particles are know (more than 100) and there are many more properties, besides spin and mass, that characterize elementary particles. Next we will classify the elementary particles by the kind of interaction which act on each particle. We know that there are four different interactions, which are the electromagnetic, strong, weak and gravitational interactions. The gravitational interaction is special in the sense that gravity will act on anything that has energy and therefore any elementary particle will interact through gravitation. But this interaction will also be very weak. The other three interactions will only act on certain elementary particles. So for example the electromagnetic interaction will act on the proton but not on the neutron.

Next consider the electron. Since it has an electromagnetic charge it will be subjected to the electromagnetic interaction. But the electron is also subjected to the weak interaction. Indeed, the standard example of a process involving the weak interaction is the so-called β-decay

$$n \to p + e^- + \bar{\nu}_e.$$

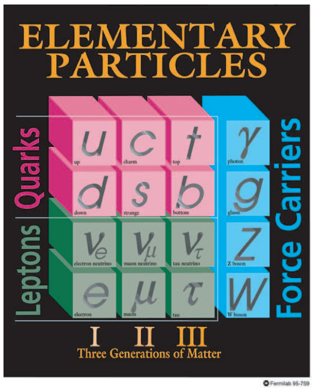
This decay of the neutron can be represented graphically as in the figure below



In the above diagram the W boson is the particle responsible for mediating the weak force. This example makes it clear that the electron is subjected to the weak interaction.

Moreover the electron does not interact hadronically. This can be seen by bombarding nuclei with electrons. The scattering can be explained by invoking the electromagnetic interaction only.

As we already discussed the known elementary particles can be constructed from the building blocks from the table below



The different forces are mediated by the so-called gauge bosons, which can be the photon, the gluon, the W or Z bosons. The building blocks of matter can be either leptons or hadrons. The leptons interact weakly, electromagnetically if they are charged but they never take part in the strong interaction. All other particles are hadrons and they are constructed from quarks. Hadrons interact weakly, strongly and electromagnetically if they are charged.

In the following we will be discussing leptons and hadrons in some more detail.

2.4 Leptons

In the table below the known leptons are summarized together with their spin, mass, magnetic moment and lifetime.

Lepton	**Spin**	**(Mass)c^2**	$\mu(e\hbar/2mc)$	**Lifetime**
ν_e	1/2	0.55 meV	0	stable
e^-	1/2	0.511 MeV	-1.001	stable
ν_μ	1/2	47.5 meV	0	stable
μ^-	1/2	105.7 MeV	-1.001	2.197 μs
ν_τ	1/2	0.54 eV	0	probably stable
τ^-	1/2	1784 MeV	?	3.3×10^{-13} s.

In this table we see that there is a fundamental difference between the muon and the electron. The electron is stable and the muon is not.

All three subatomic interactions can be responsible for the decay of particles. As a general rule (to which many exceptions exist) particles that decay as a result of the different interactions also have different lifetimes. As an example consider the following table[⋆]

Interaction	**Example**	**Lifetime**
hadronic	Δ	$10^{-23}s$
electromagnetic	π^0	$10^{-18}s$
weak	Λ	$10^{-10}s$

⋆ Note that the particles in this table are not leptons but hadrons which we will be discussing in some more detail in the next sections.

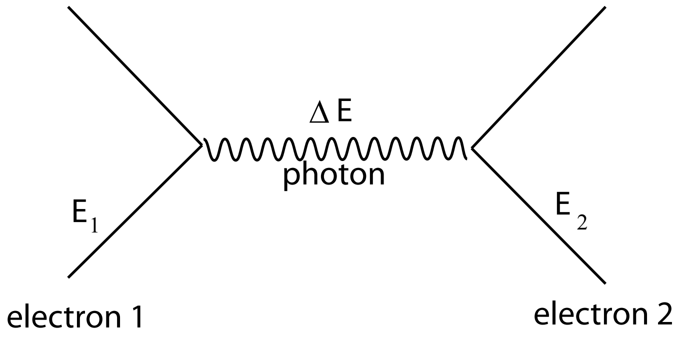
2.5 Mesons

Hadrons can be separated into mesons and baryons. In terms of the quark structure a meson would be a bound state of a quark and an anti-quark, while a baryon would be a bound state of three quarks. While many baryons are stable, like the proton, all the mesons decay through one of the three interactions that we discussed in the previous sections.

The first meson that was ever discovered was the pion π. The pion is a particle that was predicted theoretically by Yukawa in 1938. It was only 10 years after this prediction that the π could be found experimentally. In the following we will be discussing in detail how such a prediction was made since this will also illustrate the idea of mediating interactions through the exchange of particles. We will start by discussing the more familiar example of the electromagnetic force between two electrons.

We have seen when we discussed special relativity that nothing can travel faster than the speed of light c. Once we have accepted special relativity then we have to give up the that idea there is an interaction at a distance. So for example in Newton's theory of gravitation it is assumed that the interaction between two bodies is instantaneous. But according to special relativity a rapid acceleration of the sun can only affect the earth after the 8 minutes that it takes the light to travel from the sun to the earth. In a theory that combines special relativity and quantum theory we imagine that the forces are not mediated at a distance but that there is an exchange of a particles taking place.

The electromagnetic interaction between two electrons, for example, takes place through the exchange of a photon as can be seen in the picture below



In order to understand how an interaction in a quantum theory take place we will next discuss how energy conservation is handled in a quantum theory. In the previous sections we have learned that in special relativity energy and momentum are conserved. It turns out that in a quantum theory energy conservation can be violated for very small periods of time. Indeed, in a quantum theory there is an energy and time uncertainty relation which states

$$\Delta E \Delta t \geq \hbar,$$

which is similar to the uncertainty relation between the position and the momentum, $\Delta p \Delta x \geq \hbar$ that we already met.

Due to this energy and time uncertainty principle the photon that is being exchanged between the interacting electrons cannot be observed. Indeed, the photon that is being exchanged will be emitted and then absorbed again after a small time that we may call Δt. If the particle has an energy ΔE then for time intervals smaller than $\Delta t \approx \hbar/\Delta E$ the photon will not be observed. Because of this the photon is called virtual.

We can use the above reasoning to find an approximation for the range of the different interactions. Indeed, during the time interval $\Delta t \approx \hbar/\Delta E$ the virtual particle mediating the interaction will travel at most the distance $c\Delta t \approx c\hbar/\Delta E$ before being absorbed by the second electron. If the virtual particle that is being exchanged has a mass m then the energy that it will cost to generate it is at least its rest energy, namely mc^2. This means that we can approximate the range of the interaction to be

$$r_0 = \frac{c\hbar}{\Delta E} = \frac{\hbar}{mc},$$

which is exactly the Compton wave length of the exchanged particle.

In case that we are considering a particle with zero rest mass like the photon we see that the range of the interaction is infinite. That the photon is massless has been verified experimentally to a very high precision. Indeed, if the photon would have a non-vanishing mass then Coulomb's law would be corrected. No such a correction has been measured to date. This implies that the mass of the photon has to be smaller than $10^{-16}\ eV = 8 \times 10^{-49} g$.

But the situation for the strong interaction is quite different. In 1934 it was known that the hadronic interaction must be very strong and that it must have a range of the size of the atomic nucleus ($\approx 2\ fermi$, where $1\ fermi = 10^{-15} m$). This lead the Japanese physicist Yukawa to postulate that a new elementary particle is being exchanged. This particle was called the pion. Using the reasoning above the mass of this pion could be predicted. The result was

$$r_0 = \frac{\hbar}{mc} \approx 1.4\ fermi.$$

This implied that the mass of the virtual pion would have to be around 100 MeV. The pion with a mass of 135 MeV was found 10 years later.

Yukawa's theory was very important for the further understanding of the strong interactions. But at very high energies this theory does not provide the correct description of the strong interaction and has to be replaced by QCD.

The above picture representing the interaction of two electrons by exchanging a photon is an example of a Feynman diagram. As we will later see these diagrams provide an elegant procedure of calculation. Once we have drawn a diagram we can say exactly what the value of the quantum mechanical amplitude is. Or said differently, a Feynman diagram can be translated into a result for the potential between elementary particles. The diagram representing the two interacting electrons corresponds to the Coulomb potential

$$V(r) = -\frac{e^2}{r}.$$

In the same way if the particle that is being interchanged is massive then the Coulomb potential is replaced by the Yukawa potential

$$V(r) = const.\frac{e^{-r/r_0}}{r}.$$

For the weak interaction the situation is the following. The weak interaction has a very short range (approximately 10^{-18} m). Because of the relation between the range and the mass of the virtual particle that is being exchanged it becomes clear that the virtual particles responsible for the weak interaction must be very heavy. Because of their high mass (which is approximately 80 GeV/c^2 for the W's and 90 GeV/c^2 for the Z) and because they only interact weakly is has been very difficult to confirm experimentally the existence of the particles responsible for the weak interaction. The W particles were finally found in 1983 in the proton-antiproton collider SPS at CERN which was operating at an energy of 270 GeV.

For the case of gravitation we would expect the mass of the graviton to be zero since the range is infinite. But since gravity is so weak it turns out to be almost impossible to experimentally confirm the existence of gravitons. Until these days only very indirect evidence for the existence of gravitational waves has been found. These waves would be the analogue of electromagnetic waves and are the classical limit of the graviton which would appear in a quantum theory of gravity.

The discussion above describes how a new elementary particle, the pion, was predicted. But more powerful than an additional elementary particle is the concept that interactions are mediated through virtual elementary particles.