Arithmetic of Calabi-Yau Manifolds

with

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Aims:

- **To explain the fact that the periods of a Calabi–Yau manifold in terms of which we compute many observables of the effective low energy limit of string theory encode important arithmetic information about the manifold.**

- **To speculate about the role of 'quantum corrections' and mirror symmetry.**

Periods of the Quintic

Consider, for definiteness, the one parameter family of quintics in $\mathbb{P}_4$

$$\mathcal{M}\ :\ P(x,\psi)\ =\ \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1 x_2 x_3 x_4 x_5\ .$$

$\mathcal{M}$ **has** $h^{11} = 1$ **and** $h^{21} = 101$.

In this simple case there is a simple relation between $\mathcal{M}$ **and its mirror**

$$\mathcal{W}\ =\ \mathcal{M}/\Gamma$$

$$\Gamma\ :\ (x_1,\ x_2,\ x_3,\ x_4,\ x_5)\ \mapsto\ (\zeta^{n_1}x_1,\ \zeta^{n_2}x_2,\ \zeta^{n_3}x_3,\ \zeta^{n_4}x_4,\ \zeta^{n_5}x_5)$$

where $\zeta^5 = 1$ **and** $\sum_i n_i \equiv 1\ mod\ 5$.

$\mathcal{M}$ has $h^{21} = 101$ and $204 = 2 \times 100 + 4$ periods while $\mathcal{W}$ has $h^{21} = 1$ and 4 periods.

These periods are hypergeometric functions and satisfy a differential equation

$$\mathcal{L}\,\varpi(\lambda) = 0 \; ; \qquad \lambda = \frac{1}{(5\psi)^5} \, .$$

where

$$\mathcal{L} = \vartheta^4 - 5\lambda \prod_{i=1}^{4}(5\vartheta + i) \; , \quad \text{with} \;\; \vartheta = \lambda \frac{d}{d\lambda} \, .$$

The operator $\mathcal{L}$ is of fourth order and $\lambda = 0$ is a regular singular point with all four indices equal to zero. Thus the solutions near the origin are asymptotic to

$$1, \; \log\lambda, \; \log^2\lambda, \; \log^3\lambda \, .$$

The solution that has no logarithm is the series

$$f_0(\lambda) \;=\; \sum_{m=0}^{\infty} \frac{(5m)!}{(m!)^5}\, \lambda^m \;.$$

more generally the solutions are of the form

$$\begin{aligned}
\varpi_0(\lambda) \;&=\; f_0(\lambda)\\
\varpi_1(\lambda) \;&=\; f_0(\lambda)\log\lambda + f_1(\lambda)\\
\varpi_2(\lambda) \;&=\; f_0(\lambda)\log^2\lambda + 2f_1(\lambda)\log\lambda + f_2(\lambda)\\
\varpi_3(\lambda) \;&=\; f_0(\lambda)\log^3\lambda + 3f_1(\lambda)\log^2\lambda + 3f_2(\lambda)\log\lambda + f_3(\lambda)
\end{aligned}$$

where the $f_j(\lambda)$ are power series. These series will enter into our calculation of the number of rational points of $\mathcal{M}$. Recall that these solutions may be found by the method of Frobenius. That is by seeking solutions of the form

$$\varpi(\lambda,\varepsilon) \;=\; \sum_{m=0}^{\infty} a_m(\varepsilon)\, \lambda^{m+\varepsilon} \quad \text{to the equation} \quad \mathcal{L}\,\varpi(\lambda,\varepsilon) \;=\; \varepsilon^4 \lambda^\varepsilon \;.$$

We can fix a basis by choosing $a_0(\varepsilon) = 1$, independent of ε,

$$a_m(\varepsilon) \;=\; \frac{\Gamma(1+\varepsilon)^5}{\Gamma(1+5\varepsilon)}\,\frac{\Gamma(1+5m+5\varepsilon)}{\Gamma(1+m+\varepsilon)^5}$$

We may however replace the a_m by another choice. This freedom corresponds to multiplying $\varpi(\boldsymbol{\lambda},\varepsilon)$ by a function $h(\varepsilon)$ with $h(0) = 1$ *i.e.* solving the equation $\mathcal{L}\widetilde{\varpi}(\boldsymbol{\lambda},\varepsilon) = h(\varepsilon)\varepsilon^4$.

$$\widetilde{a}_m(\varepsilon) \;=\; h(\varepsilon)\,a_m(\varepsilon) \quad \text{with} \quad h(\varepsilon) \;=\; \sum_{k=0}^{\infty}\frac{1}{k!}\,h_k\,\varepsilon^k\,.$$

This yields another basis:

$$\begin{pmatrix}\widetilde{f}_0\\ \widetilde{f}_1\\ \widetilde{f}_2\\ \widetilde{f}_3\\ \widetilde{f}_4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ h_1 & 1 & 0 & 0 & 0\\ h_2 & 2h_1 & 1 & 0 & 0\\ h_3 & 3h_2 & 3h_1 & 1 & 0\\ h_4 & 4h_3 & 6h_2 & 4h_1 & 1\end{pmatrix}\begin{pmatrix}f_0\\ f_1\\ f_2\\ f_3\\ f_4\end{pmatrix}$$

Integral Series

We know what the integers mean for the q-expansion of the yukawa coupling:

$$y_{ttt} = 5\left(\frac{2\pi i}{5}\right)^3 \frac{\psi^2}{\varpi_0(\psi)^2(1-\psi^5)}\left(\frac{d\psi}{dt}\right)^3 = 5 + \sum_{k=0}^{\infty}\frac{n_k k^3 q^k}{1-q^k}\,,$$

where in this expression

$$q = \exp(2\pi i t) \quad \text{and} \quad t = \frac{1}{2\pi i}\frac{\varpi_1(\lambda)}{\varpi_0(\lambda)}\,.$$

Integers however appear also in the mirror map

$$\begin{aligned}\lambda = q &+ 154\,q^2 + 179139\,q^3 + 313195944\,q^4 \\ &+ 657313805125\,q^5 + 1531113959577750\,q^6 \\ &+ 3815672803541261385\,q^7 \\ &+ 9970002717955633142112\,q^8 + \ldots\,.\end{aligned}$$

Rational Points

Now ask a very strange question:

For the quintic $\mathcal{M}$

$$P(x,\psi) \;=\; \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1x_2x_3x_4x_5$$

how many solutions of the equation $P(x,\psi) = 0$ are there with integer x_i and how does this number vary with ψ?

Since the x_i are coordinates in a projective space and we are free to multiply the coordinates by a common scale there is no difference between seeking an integral solution and a rational solution, $x_i \in \mathbb{Q}$. This formulation is better because $\mathbb{Q}$ is a field but it is still very hard to answer in general. An easier but still interesting question is how many solutions are there over a finite field.

Counting the Number of Points Exactly

Denote by ν_λ the number of solutions to the equation $P(x,\psi)=0$ over $\mathbb{F}_p$.

$$\nu_\lambda = {}^p f_0(\Lambda) + \left(\frac{p}{1-p}\right) {}^p f_1'(\Lambda) + \frac{1}{2!}\left(\frac{p}{1-p}\right)^2 {}^p f_2''(\Lambda)$$
$$+ \frac{1}{3!}\left(\frac{p}{1-p}\right)^3 {}^p f_3'''(\Lambda) + \frac{1}{4!}\left(\frac{p}{1-p}\right)^4 {}^p f_4''''(\Lambda) + \mathcal{O}(p^5) \; .$$

This expression holds for $5 \nmid p-1$. In the expression

$$\Lambda \;=\; \text{Teich}(\lambda) \;=\; \lim_{n\to\infty} \lambda^{p^n} \quad \text{and} \quad {}^p f_0(\Lambda) \;=\; \sum_{m=0}^{p-1} \frac{(5m)!}{(m!)^5}\,\Lambda^m$$

and we have changed basis with a function $h(\varepsilon)$ with

$$h(rp) \;=\; \frac{a_{rp}}{a_r} \;=\; \frac{\Gamma_p(1+5rp)}{\Gamma_p(1+rp)^5} \; .$$

Now, as we have said, the number of rational points is determined by the periods and there are $b^3 = 2h^{21} + 2$ of these. The Hodge number h^{21} counts the number of parameters on which the complex structure depends and, in simple cases, this corresponds to the number of ways of deforming the defining polynomial

$$P(x,c) = \sum_{\vec{v}} c_{\vec{v}}\, x^{\vec{v}} \quad ; \quad x^{\vec{v}} = x_1^{v_1}\, x_2^{v_2}\, x_3^{v_3}\, x_4^{v_4}\, x_5^{v_5}\ .$$

The directions in which $P(x,c)$ can be deformed correspond to the monomials $x^{\vec{v}}$ considered subject to the ideal $(\partial P/\partial x_i)$. A special role is played by fundamental monomial

$$Q = x_1 x_2 x_3 x_4 x_5$$

which is related by mirror symmetry to the Kähler form of the mirror.

Return now to our special one parameter family of polynomials

$$P(x,\psi) = \sum_{i=1}^{5} x_i^5 - 5\psi\, x_1x_2x_3x_4x_5 \,.$$

$\mathcal{M}$ has $2h^{21}(\mathcal{M}) + 2 = 204 = 2\times 100 + 4$ periods while $\mathcal{W}$ has $2h^{21}(\mathcal{W}) + 2 = 4$.

$$1 \longrightarrow Q \longrightarrow Q^2 \longrightarrow Q^3$$

$$x^v \longrightarrow Q\, x^v$$

This leads to 1 fourth order differential operator $\mathcal{L}_{\vec{1}}$ and 100 second order operators $\mathcal{L}_{\vec{v}}$.

There are tenth order monomials that are not included in the above scheme and which require special attention. The generators of the ideal are

$$x_1^4 \simeq \psi\, x_2x_3x_4x_5 \ \text{ \& cyclic.}$$

Thus

$$x^{(4,3,2,1,0)} \simeq \psi\, x^{(0,4,3,2,1)} \simeq \cdots \simeq \psi^5\, x^{(4,3,2,1,0)} \,.$$

We can also perform the sum in our expression for the number of points to give

$$\nu_\lambda = \sum_{m=0}^{p-1} \beta_m \Lambda^m$$

with coefficients

$$\beta_m = \lim_{n\to\infty} \frac{a_{m(1+p+p^2+\ldots+p^{n+1})}}{a_{m(1+p+p^2+\ldots+p^n)}} = (-1)^m G_{5m} G^5_{-m}$$

When we include the contributions of the other periods for the case $5|p-1$ we find

$$p\nu^*_\lambda = (p-1)^5 + \sum_{\vec{v}} \sum_{m=0}^{p-2} (-1)^m \Lambda^m G_{5m} \prod_{j=1}^{5} G_{-(m+kv_j)}$$

where $k = (p-1)/5$. The contribution of $\vec{v} = (0,0,0,0,0)$ gives our previous expression. The quintic $\vec{v}$'s correspond to the other 200 periods and give the extra terms that arise when $5|p-1$. These terms have a natural interpretation as the exceptional divisors of the mirror manifold. The monomial of degree 10 contributes only for the conifold when $\psi^5 = 1$.

The Zeta-Function

Consider now $N_r(\lambda) = \frac{\nu_\lambda - 1}{p-1}$ which are the numbers of projective solutions of $P = 0$ over $\mathbb{F}_{p^r}$ and form

$$\zeta(T,\lambda) = \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\lambda)\, T^r}{r}\right) .$$

If $\mathcal{M}$ is a point then $N_r = 1$ for all r and

$$\sum_{r=1}^{\infty} \frac{N_r T^r}{r} = \sum_{r=1}^{\infty} \frac{T^r}{r} = -\log(1-T) \implies \zeta_{\text{pt}}(T) = \frac{1}{1-T}$$

Thus for a point

$$\prod_p \zeta_{\text{pt}}(p^{-s}) = \prod_p \frac{1}{1-p^{-s}} = \zeta_R(s) .$$

The Weil Conjectures

- **Rationality** **(Dwork):** $\zeta(T)$ **is a rational function of** T

- **Functional equation** **(Groethendieck):**

$$\zeta\left(\frac{1}{p^d T}\right) = \pm p^{d\chi/2}\, T^{\chi}\, \zeta(T)$$

where χ **is the Euler characteristic and** d **is the real dimension of** $\mathcal{M}$**.**

- **Riemann Hypothesis** **(Deligne):**

$$\zeta(T) = \frac{P_1(T)P_3(T)\ldots P_{2d-1}(T)}{P_0(T)P_2(T)\ldots P_{2d}(T)}$$

with $P_i(T)$ **a polynomial with coefficients in** $\mathbb{Z}$ **of degree** b_i**. Furthermore**

$$P_i(T) = \prod_{j=1}^{b_i}(1-\alpha_{ij}\,T)\,,\ \ |\alpha_{ij}| = p^{i/2} \ \text{ and } \ P_0(T) = 1-T\,,\ \ P_{2d}(T) = 1-p^d T\,.$$

The ζ-Function

We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x,\psi) = 0$. The ζ-function is defined by the expression

$$\zeta(t,\psi) \;=\; \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\psi)t^r}{r}\right)$$

We are led to decompose N_r into a sum of contributions $N_r \;=\; N_{r,0} + \sum_v N_{r,v}$.

$$\zeta_{\mathcal{M}}(t,\psi) \;=\; \frac{R_0(t,\psi)\;\prod_v R_v(t,\psi)}{(1-t)(1-pt)(1-p^2t)(1-p^3t)}$$

$$\zeta_{\mathcal{W}}(t,\psi) \;=\; \frac{R_0(t,\psi)}{(1-t)(1-pt)^{101}(1-p^2t)^{101}(1-p^3t)} \; .$$

In all cases, apart from the conifold, R_0 is a quartic

$$R_0 \;=\; 1 + a_0\,t + b_0\,pt^2 + a_0\,p^3t^3 + p^6t^4 \; .$$

The Euler Curves

Classical analysis gives an expression for the hypergeometric functions in terms of Euler's integral which is of the form

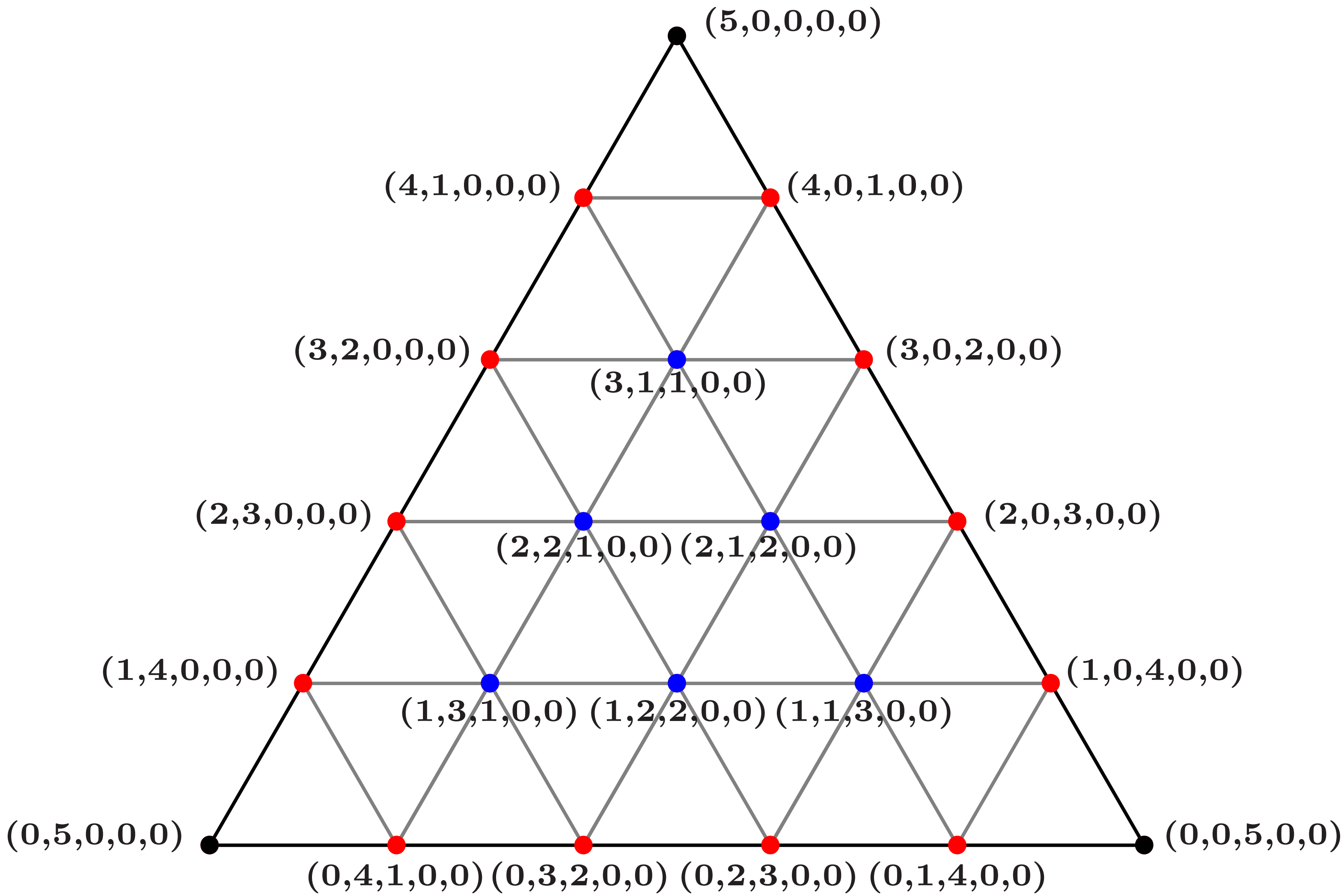
$$\int dx\, x^{-\alpha/5}(1-x)^{-\beta/5}(1-x/\psi^5)^{-(1-\beta/5)}\ .$$

If we think of Euler's integral as $\int \frac{dx}{y}$ **then we are led to curves**

$$\mathcal{E}_{\alpha\beta}(\psi):\quad y^5 = x^\alpha(1-x)^\beta(1-x/\psi^5)^{5-\beta}\ .$$

v	α	β
$(4,1,0,0,0)$	2	3
$(3,2,0,0,0)$	1	4
$(3,1,1,0,0)$	2	4
$(2,2,1,0,0)$	4	3

$$\mathcal{E}_{\alpha\beta} = \begin{cases} \mathcal{A} & \alpha+\beta=5 \\ \mathcal{B} & \alpha+\beta\neq 5 \text{ and } \alpha\neq\beta\ . \end{cases}$$


(5,0,0,0,0)
(4,1,0,0,0)
(4,0,1,0,0)
(3,2,0,0,0)
(3,1,1,0,0)
(3,0,2,0,0)
(2,3,0,0,0)
(2,2,1,0,0)
(2,1,2,0,0)
(2,0,3,0,0)
(1,4,0,0,0)
(1,3,1,0,0)
(1,2,2,0,0)
(1,1,3,0,0)
(1,0,4,0,0)
(0,5,0,0,0)
(0,4,1,0,0)
(0,3,2,0,0)
(0,2,3,0,0)
(0,1,4,0,0)
(0,0,5,0,0)

For the curve $\mathcal{A}$ there is a corresponding ζ-function

$$\zeta_{\mathcal{A}}(t) = \frac{R_{\mathcal{A}}(t)^2}{(1-t)(1-pt)} .$$

Now the existence of nontrivial fifth roots of unity is important for the mirror construction. Such roots of unity exist in $\mathbb{F}_{p^r}$ precisely when $5|p^r - 1$. For given p let $\rho = 1, 2$ or 4 be the smallest r for which $5|p^r - 1$.

The $R_{\vec{v}}$ pair up in the following way:

$$\begin{aligned} R_{(4,1,0,0,0)}(t)\, R_{(3,2,0,0,0)}(t) &= R_{\mathcal{A}}(p^\rho t^\rho)^{1/\rho} \\ R_{(3,1,1,0,0)}(t)\, R_{(2,2,1,0,0)}(t) &= R_{\mathcal{B}}(p^\rho t^\rho)^{1/\rho} . \end{aligned}$$

So the ζ-function for $\mathcal{M}$ takes the form

$$\zeta_{\mathcal{M}}(t,\psi) = \frac{R_0(t,\psi)\, R_{\mathcal{A}}(p^\rho t^\rho,\psi)^{\frac{30}{\rho}}\, R_{\mathcal{B}}(p^\rho t^\rho,\psi)^{\frac{20}{\rho}}}{(1-t)(1-pt)(1-p^2t)(1-p^3t)} .$$

The Z-Function and Mirror Symmetry

We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x,\psi) = 0$.

$$\zeta(T,\psi) \;=\; \exp\left(\sum_{r=1}^{\infty} \frac{N_r(\psi)T^r}{r}\right)$$

As defined the ζ-function does not respect mirror symmetry

$$\zeta(T) \;=\; \frac{\text{Numerator of deg. } 2h^{21}+2 \text{ depending on the cpx. structure of } \mathcal{M}}{\text{Denominator of deg. } 2h^{11}+2}\,.$$

Explicitly for the quintic we have

$$\zeta_{\mathcal{M}}(T,\psi) \;=\; \frac{R_0(T,\psi)\,R_{\mathcal{A}}(p^\rho T^\rho,\psi)^{\frac{20}{\rho}}\,R_{\mathcal{B}}(p^\rho T^\rho,\psi)^{\frac{30}{\rho}}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}$$

$$\zeta_{\mathcal{W}}(T,\psi) \;=\; \frac{R_0(T,\psi)}{(1-T)(1-pT)^{101}(1-p^2T)^{101}(1-p^3T)}$$

The Conifold

For the conifold $\psi^5 = 1$ the ζ-function seems to be especially simple

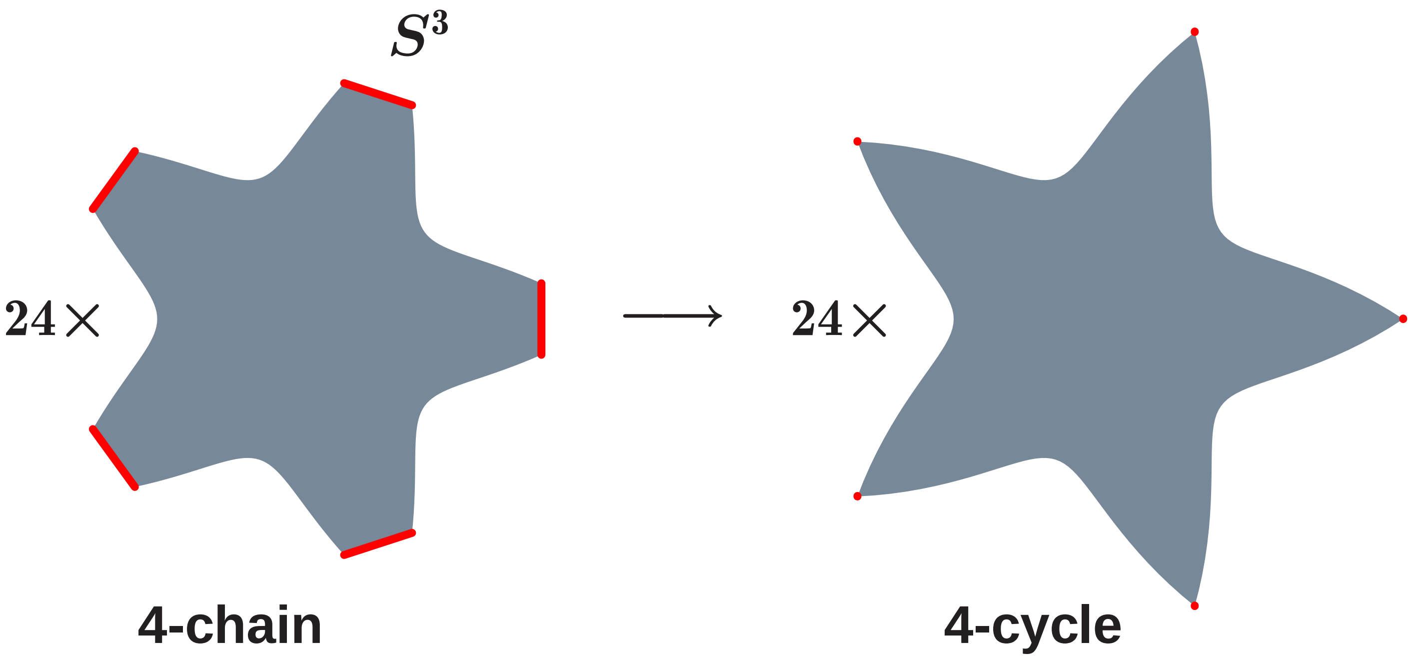
$$\zeta(T,1) \;=\; \frac{(1-\epsilon\, pT)\,(1-a_p\,T+p^3T^2)\,(1-pT)^{100}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)\,(1-p^2T)^{24}} \quad ; \quad \rho = 1$$

where $\epsilon = \left(\frac{5}{p}\right) = \pm 1$ and a_p is the p-th coefficient in the q-expansion of the eigenform, g, found by Schoen; it is the unique cusp form of weight 4 for the group $\Gamma_0(25)$.

$$g = \eta(q^5)^4\left[\eta(q)^4 + 5\eta(q)^3\eta(q^{25}) + 20\eta(q)^2\eta(q^{25})^2 + 25\eta(q)\eta(q^{25})^3 + 25\eta(q^{25})^4\right]$$

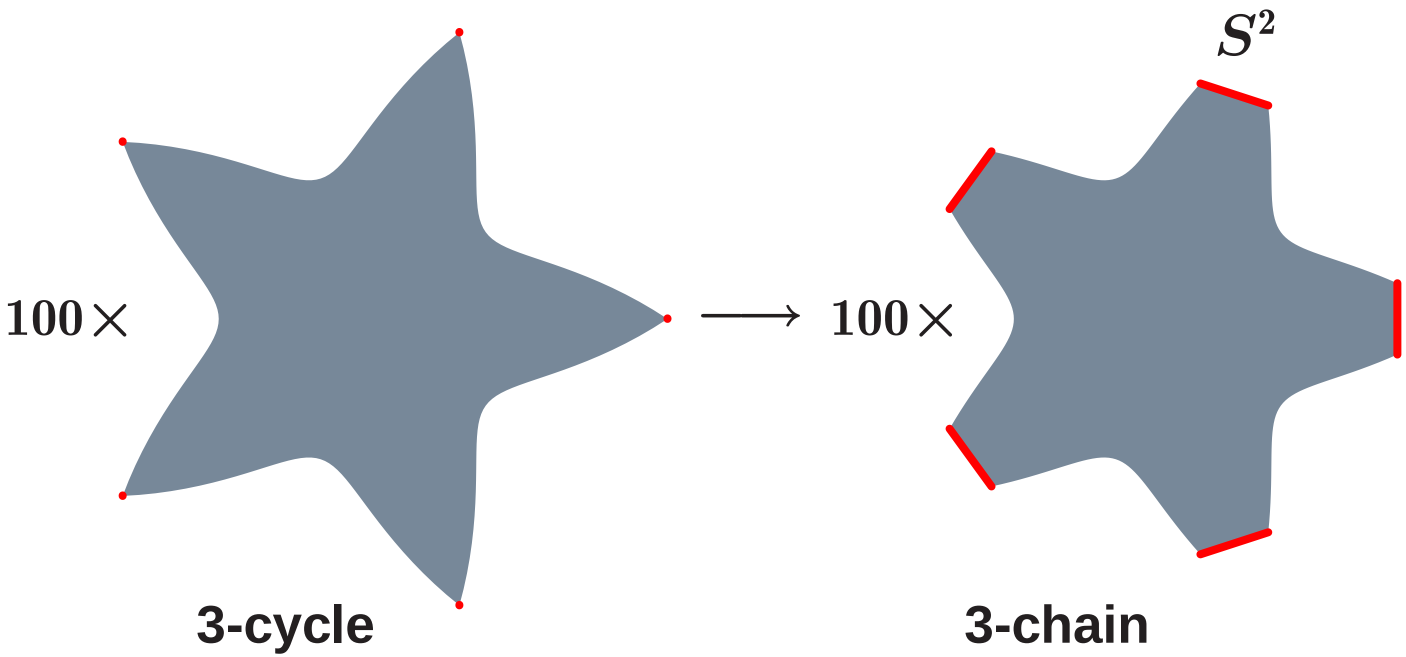
$$\begin{aligned}
&= q + q^2 + 7\,q^3 - 7\,q^4 + 7\,q^6 + 6\,q^7 - 15\,q^8 + 22\,q^9 - 43\,q^{11} - 49\,q^{12}\\
&\quad - 28\,q^{13} + 6\,q^{14} + 41\,q^{16} + 91\,q^{17} + 22\,q^{18} - 35\,q^{19} + 42\,q^{21} - 43\,q^{22}\\
&\quad + 162\,q^{23} - 105\,q^{24} - 28\,q^{26} - 35\,q^{27} - 42\,q^{28} + 160\,q^{29} + 42\,q^{31} + \cdots
\end{aligned}$$

125 S^3's are blown down but only 101 are independent so 24 4-cycles are created.



$$\zeta(T,1) \;=\; \frac{(1-a_p\,T+p^3T^2)\,(1-pT)^{100}}{(1-T)(1-p^2T)^{25}(1-p^3T)}$$

Now we resolve 125 nodes with $\mathbb{P}^1$'s, but there are 100 relations so we destroy 100 3-cycles.



$$\zeta(T,1) = \frac{(1 - a_p\, T + p^3 T^2)\,(1 - pT)^{100}}{(1 - T)(1 - pT)^{125}(1 - p^2 T)^{25}(1 - p^3 T)}$$

$$= \frac{(1 - a_p\, T + p^3 T^2)}{(1 - T)(1 - pT)^{25}(1 - p^2 T)^{25}(1 - p^3 T)}\,.$$

The Z-Function and Mirror Symmetry

We now work over $\mathbb{F}_{p^r}$ and let $N_r(\psi)$ denote the number of projective solutions to $P(x,\psi) = 0$.

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As defined the ζ-function does not respect mirror symmetry

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Explicitly for the quintic we have

$$\zeta_{\mathcal{M}}(T,\psi) \;=\; \frac{R_0(T,\psi)\, R_{\mathcal{A}}(p^\rho T^\rho,\psi)^{\frac{20}{\rho}}\; R_{\mathcal{B}}(p^\rho T^\rho,\psi)^{\frac{30}{\rho}}}{(1-T)(1-pT)(1-p^2T)(1-p^3T)}$$

$$\zeta_{\mathcal{W}}(T,\psi) \;=\; \frac{R_0(T,\psi)}{(1-T)(1-pT)^{101}(1-p^2T)^{101}(1-p^3T)}$$

The 5-adic Limit

The desired relations are true in the 5-adic limit. More precisely for all p and ψ

$$R_0(T,\psi) = (1-T)(1-p\,T)(1-p^2T)(1-p^3T) + \mathcal{O}\big(5^2\big)$$

$$R_{\mathcal{A}}(T,\psi)^{20} R_{\mathcal{B}}(T,\psi)^{30} = (1-p\,T)^{100}(1-p^2T)^{100} + \mathcal{O}\big(5^2\big)$$

so that

$$\zeta_{\mathcal{W}} = \frac{1}{\zeta_{\mathcal{M}}} + \mathcal{O}\big(5^2\big)$$

Compare this with the quantum corrections to the classical Yukawa coupling which we write in the form

$$\frac{y_{ttt}}{y_{ttt}^{(0)}} = 1 + \frac{1}{5}\sum_{k=0}^{\infty} \frac{n_k k^3 q^k}{1-q^k} = 1 + \mathcal{O}\big(5^2\big)$$

since Lian and Yau have shown that $5^3 \big| n_k k^3$ for each k.