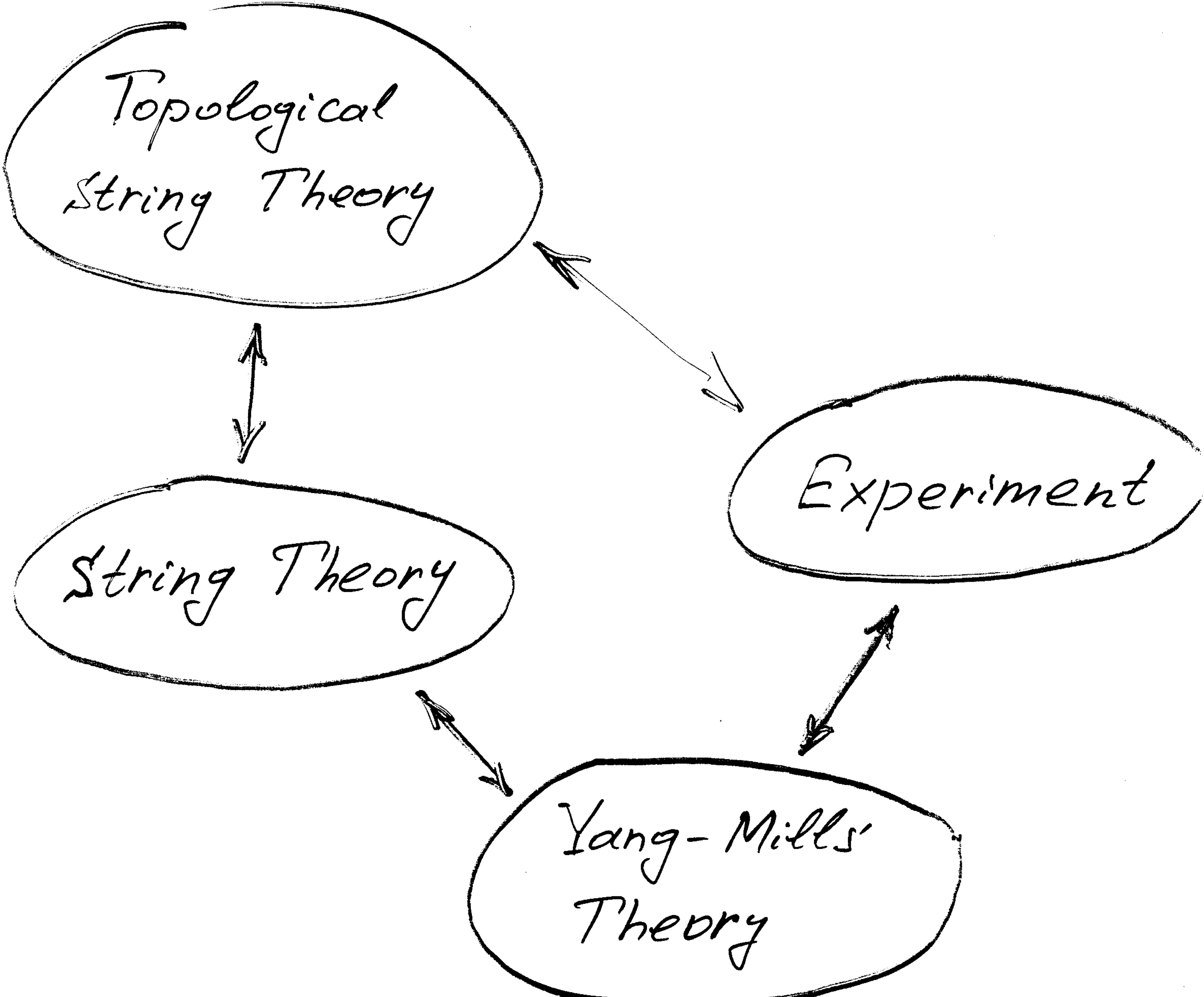
Yang-Mills Theory

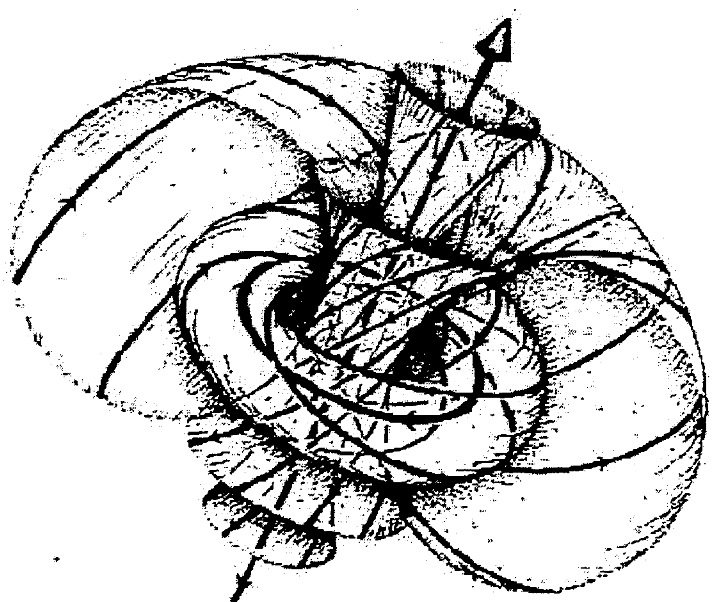
and

Moduli Spaces of Curves

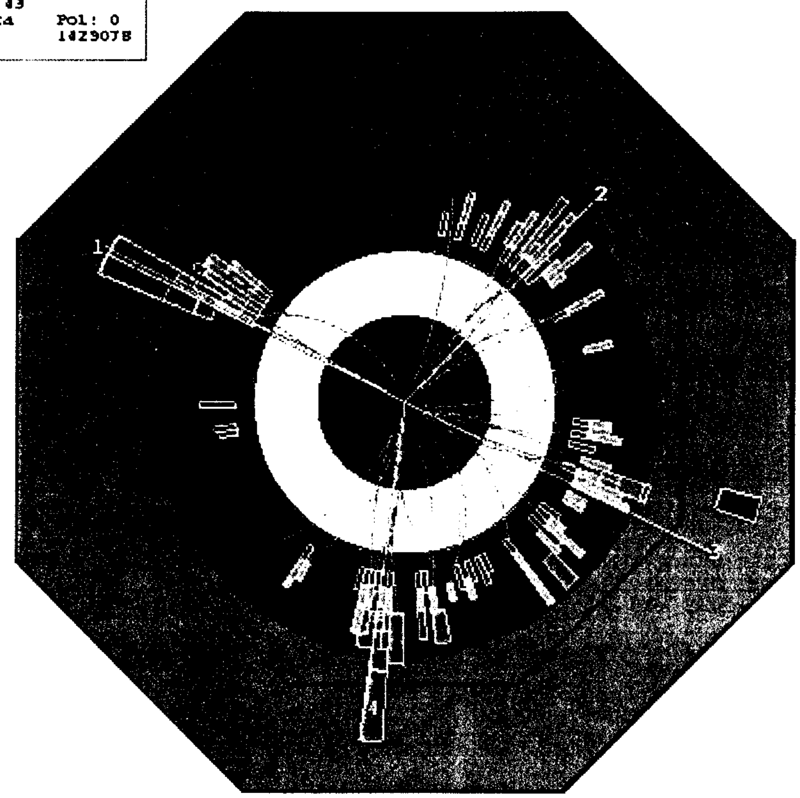

Topological String Theory
String Theory
Experiment
Yang-Mills' Theory

Outline:

- A Brief Introduction to Twistor Theory
- Yang-Mills Scattering Amplitudes
- Dual Topological String Theory (integrals over moduli spaces of curves)



Run 1536, EVENT 2126
6-JUL-1991 14:43
Source: Run Data Pol: 0
Beam Crossing 1429078



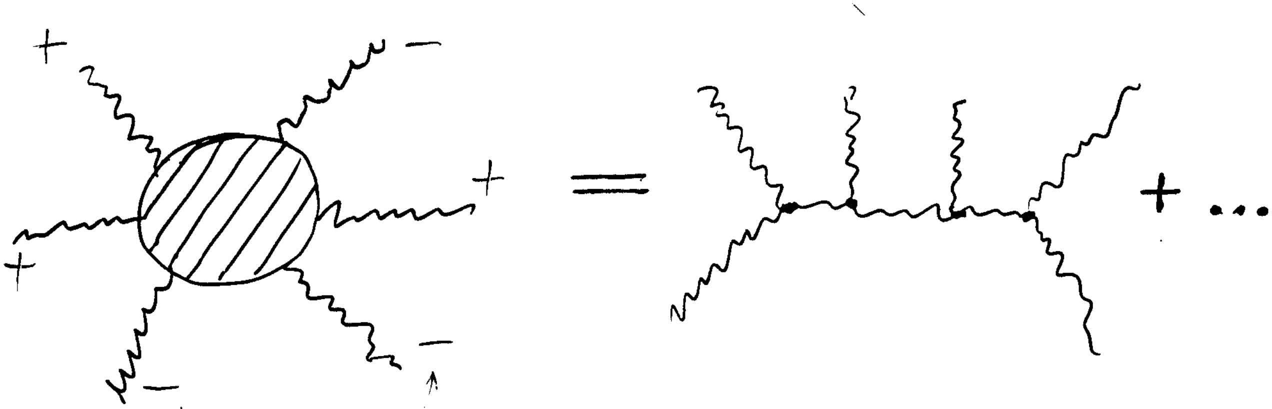
"4-jet event"

Z^0

Scattering Amplitudes:

$\mathcal{A}(+-+--+)$

Feynman diagrams:

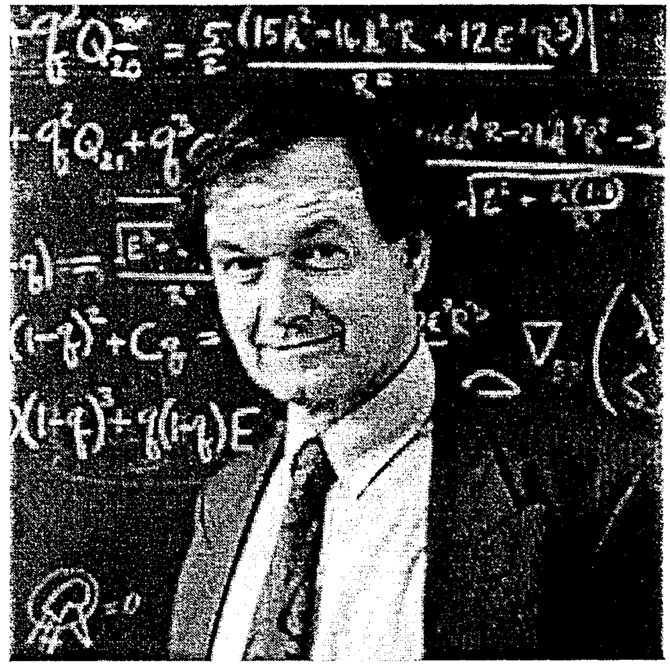


helicity

A Brief Introduction to Twistor Theory

Lorentz group

$$SO(3,1) \otimes \mathbb{C} \simeq SL(2) \times SL(2)$$



vector

$$\left(\tfrac{1}{2}, \tfrac{1}{2}\right) = \left(\tfrac{1}{2}, 0\right) \otimes \left(0, \tfrac{1}{2}\right)$$

$$\lambda^{a} \qquad \mu^{\dot{a}}$$

$$a, \dot{a} = 1, 2$$

- Twistor [illegible]:

$$Z^{A} = (\lambda^{a}, \mu^{\dot{a}})$$

- Twistor Space:

$$\mathbb{T} = \{ Z^{A} \} \simeq \mathbb{C}^{4}$$

- Projective Twistor Space:

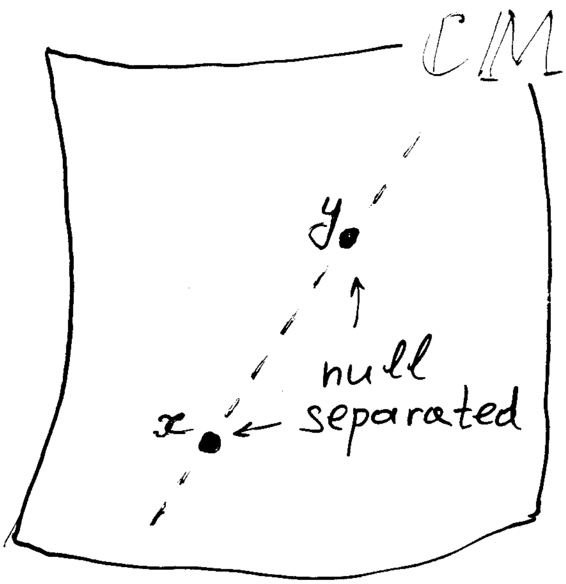
$$\mathbb{PT} \simeq \mathbb{CP}^{3}$$

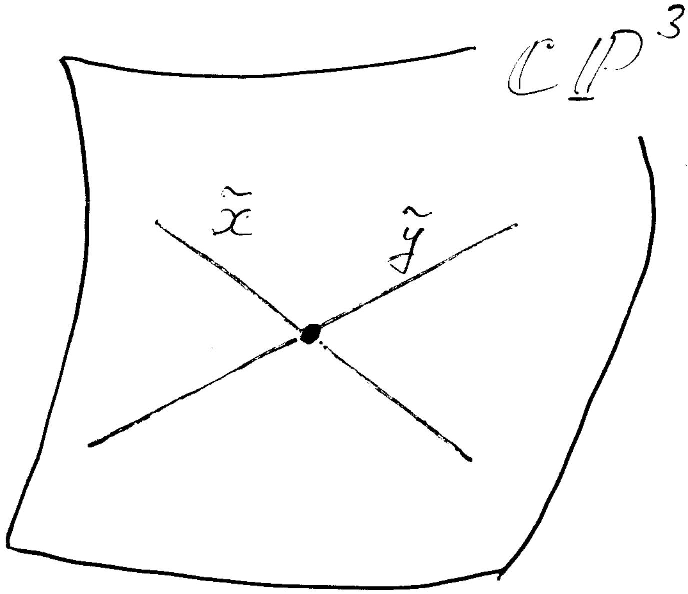
$$(\lambda^{a}, \mu^{\dot{a}}) \sim (t\lambda^{a}, t\mu^{\dot{a}}) \qquad t \in \mathbb{C}^{*}$$

Complexified Minkowski Space	Projective Twistor Space
$\mathbb{CM}$	$\mathbb{PT} = \mathbb{CP}^3$
point $x^{a\dot{a}}$ $\longleftrightarrow$	$\mathbb{CP}^1$

$$\boxed{\begin{array}{c}\text{Incidence relation}\\ \mu^{\dot{a}} + x^{a\dot{a}}\lambda_a = 0\end{array}}$$

α-plane $\longleftrightarrow$	point in $\mathbb{PT}^+$
β-plane $\longleftrightarrow$	point in $\mathbb{PT}^-$





Solutions to massless wave equations in Minkowski space	$\Longleftrightarrow$	Holomorphic data in twistor space

- tensor fields in $\mathbb{CM}$

$$\phi_{a_1 \ldots a_n} \qquad\qquad \phi_{\dot{a}_1 \ldots \dot{a}_n}$$

helicity $h = -\frac{n}{2}$ helicity $h = +\frac{n}{2}$

- helicity h

$$\nabla^2 \phi_{(h)} = 0 \longleftrightarrow H^1_{\bar\partial}\left(\mathbb{PT}', \mathcal{O}(2h-2)\right)$$

Helicity h	Massless Wave Equation	
0	Klein-Gordon equation	$\Box \phi = 0$
$-\frac{1}{2}$	Dirac equation	$\nabla^{a\dot{a}} \nu_a = 0$
$+\frac{1}{2}$		$\nabla^{a\dot{a}} \tilde{\nu}_{\dot{a}} = 0$
-1	anti-selfdual Maxwell equation	$\nabla^{a\dot{a}} F_{ab} = 0$
$+1$	selfdual Maxwell equation	$\nabla^{a\dot{a}} \tilde{F}_{\dot{a}\dot{b}} = 0$

Witten:

perturbative $\mathcal{N}=4$ super Yang-Mills	$\Longleftrightarrow$	Topological String (B-model) in twistor space $\mathbb{CP}^{3\|4}$

- $Z^A := (\lambda^1, \lambda^2, \mu^1, \mu^2 \,|\, \psi^1, \psi^2, \psi^3, \psi^4) \in \mathbb{C}^{4|4}$

holomorphic $4|4$ - form :

$$\Omega = d\lambda^1\, d\lambda^2\, d\mu^1\, d\mu^2\, d\psi^1\, d\psi^2\, d\psi^3\, d\psi^4$$

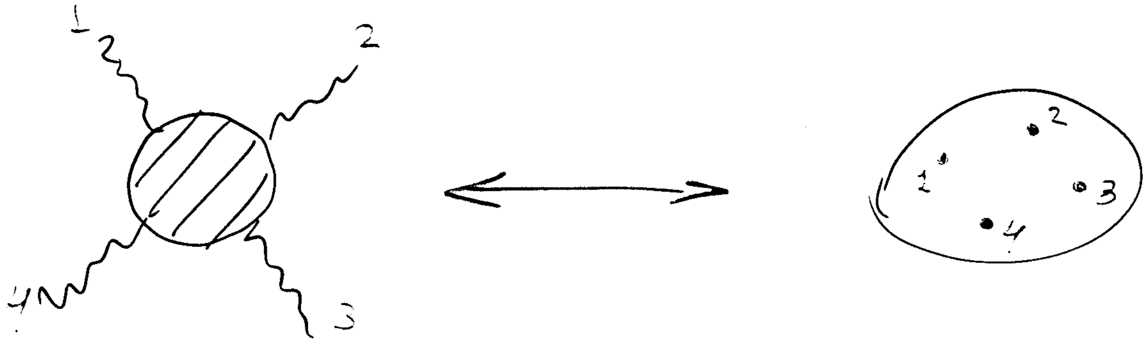
identification :

$$(\lambda, \mu \,|\, \psi) \sim (t\lambda, t\mu \,|\, t\psi) \qquad t \in \mathbb{C}^*$$

$\Longrightarrow$ Calabi-Yau supermanifold $\mathbb{CP}^{3|4}$

$$\mathcal{A}(\underbrace{+++\overbrace{---}}_{n}) = \int \omega$$

moduli space of hol. curves
in $\mathbb{CP}^{3|4}$ of degree $d = q - 1$
with n marked points

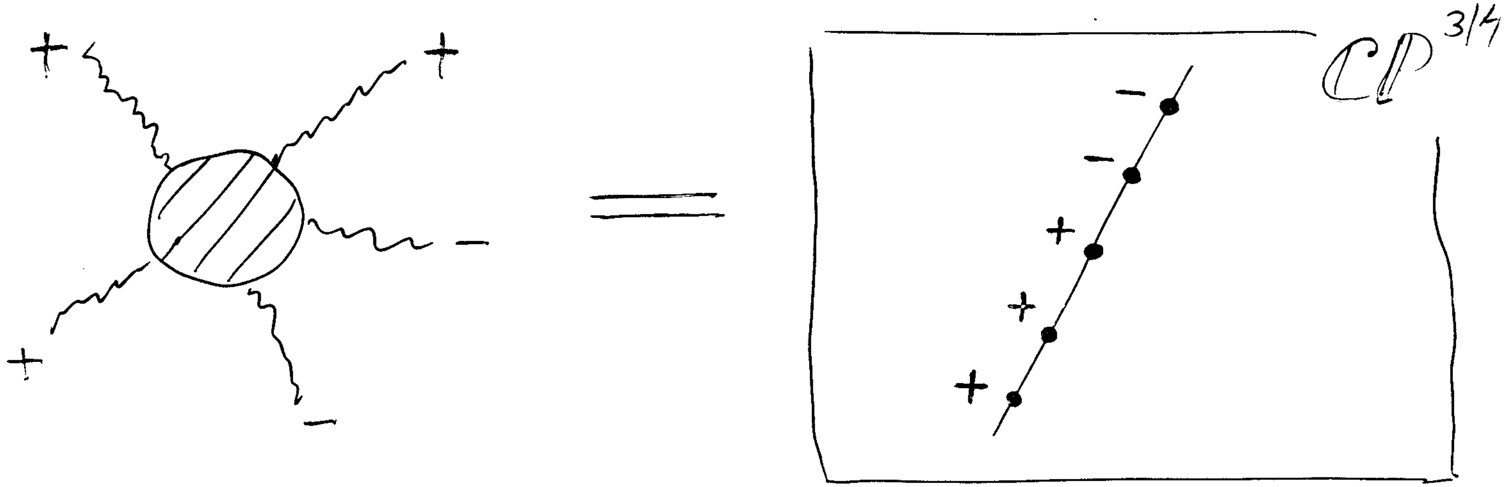


<u>**Remark:**</u> in the B-model, hol. curves are interpreted as D1-brane instantons.

Example 1:

$\mathcal{A}(+++--)$

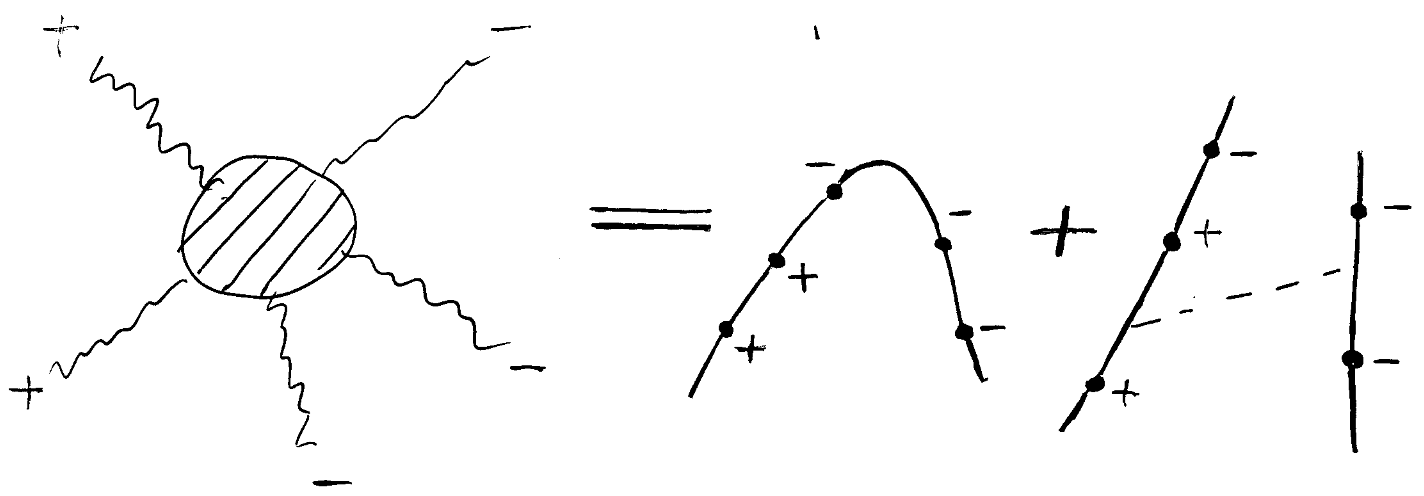
$d = q - 1 = 1$



Example 2:

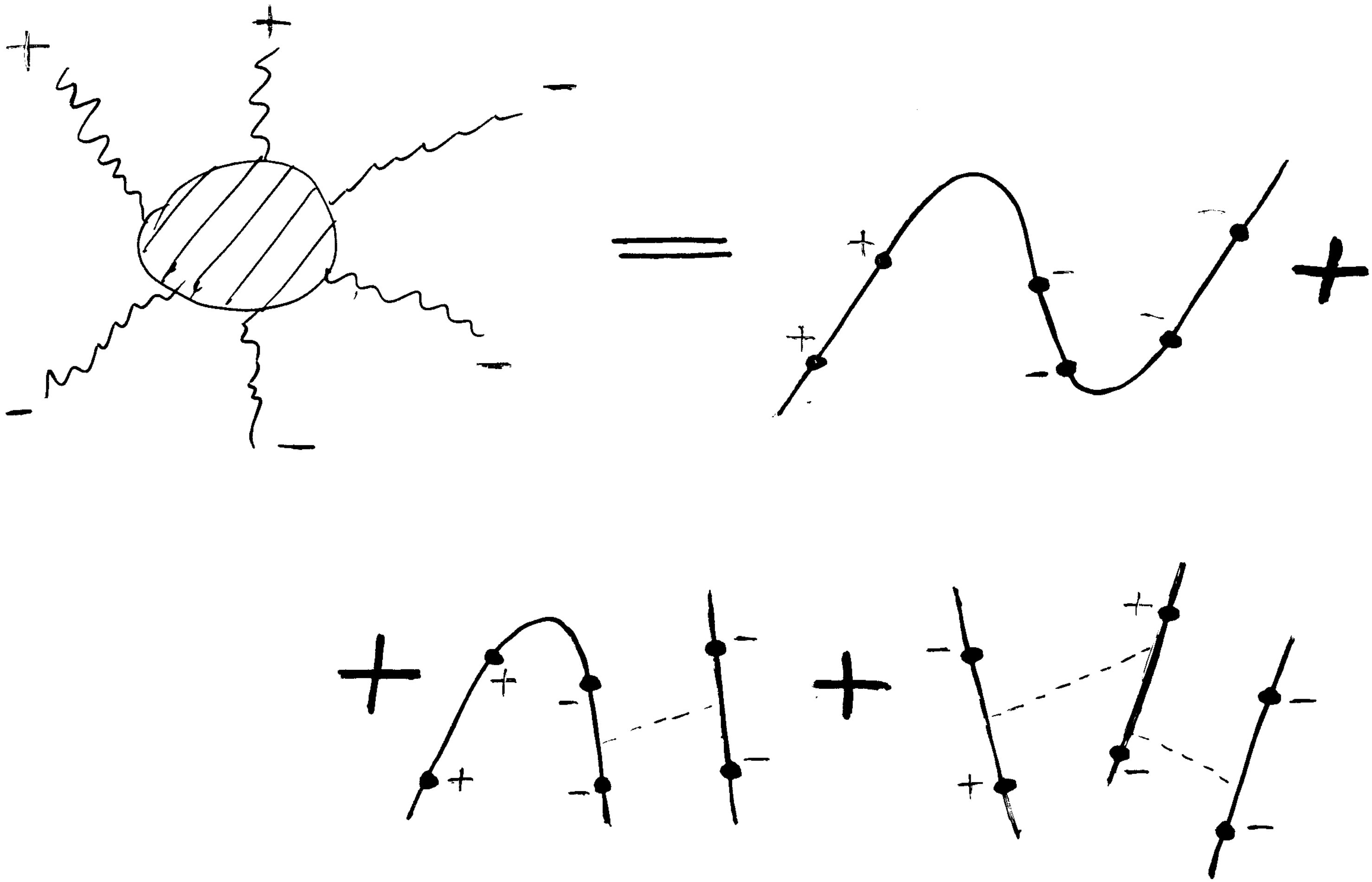
$\mathcal{A}(++---)$

$d = 2$



Example 3:

$\mathcal{A}(++----)$ $\qquad$ $\underline{d=3}$



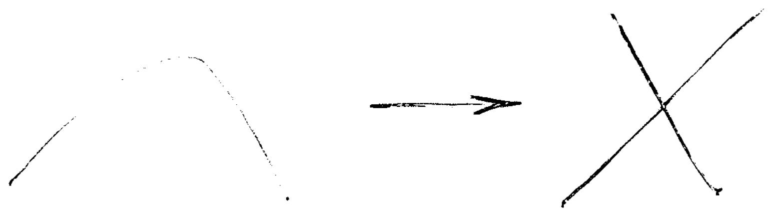
Connected Instantons:

- $\mathcal{M}_{0,n,d}(\mathbb{CP}^{3|4})$ moduli space of genus 0, n-pointed curves of degree d in $\mathbb{CP}^{3|4}$

- $\dim \mathcal{M}_{0,n,d}(\mathbb{CP}^{3|4}) = (4d+n)|(4d+4)$

- Following [[illegible]], we realize it as the space of automorphism classes of maps
$$P: \mathbb{CP}^1 \rightarrow \mathbb{CP}^{3|4}$$

- $\mathcal{M}_{0,n,d}(\mathbb{CP}^{3|4})$ is non-compact due to deg. curves, e.g.



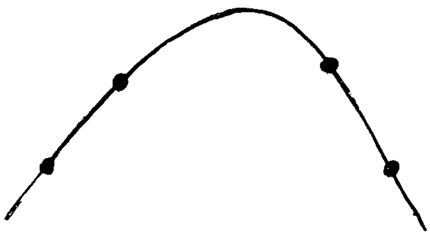
$\Rightarrow$ moduli space of stable maps
$$\overline{\mathcal{M}}_{0,n,d}(\mathbb{CP}^{3|4})$$

$$\begin{cases} P: \mathbb{CP}^1 \to \mathbb{CP}^{3|4} \\ P^{\mathcal{A}}(\sigma) = Z^{\mathcal{A}} = \displaystyle\sum_{k=0}^{d} \beta_k^{\mathcal{A}} \sigma^k \end{cases}$$

<u>moduli $\beta_k^{\mathcal{A}} \in \mathbb{C}^{4d+4|4d+4}$</u>

natural measure

$$\mu_d = \prod_{k,\mathcal{A}} d\beta_k^{\mathcal{A}}$$



$$\mathcal{A}^{conn}_{YM} = \int_{\overline{\mathcal{M}}_{0,n,d}} \prod_{k,\mathcal{A}} d\beta_k^{\mathcal{A}} \cdot \underbrace{\prod_{i=1}^{n} \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}}_{\substack{\text{hol. } n\text{-form on } (\mathbb{CP}^1)^n \\ \omega(\sigma_i)}} \prod_{i=1}^{n} ev_i^*(\phi_i)$$

- ϕ_i $\bar{\partial}$-closed $(0,1)$-form on $\mathbb{CP}^{3|4}$

• $GL(2,\mathbb{C})$: $\quad \sigma \mapsto \sigma' = \dfrac{a\sigma + b}{c\sigma + d}$

$$ad - bc \neq 0$$

$$\Rightarrow \mathcal{A}_{YM}^{conn} = \int_{\overline{\mathcal{M}}_{0,n,d}} \frac{\prod_{k,\mathbb{A}} dB_k^{\mathbb{A}} \prod_{i=1}^{n} \frac{d\sigma_i}{\sigma_i - \sigma_{i+1}}}{GL(2,\mathbb{C})} \wedge \prod_{i=1}^{n} ev_i^*(\phi_i)$$

$$= \int_{\overline{\mathcal{M}}_{0,n,d}} \frac{\mu_d \wedge \omega(\sigma_1, \dots, \sigma_n)}{GL(2,\mathbb{C})} \wedge \Phi$$

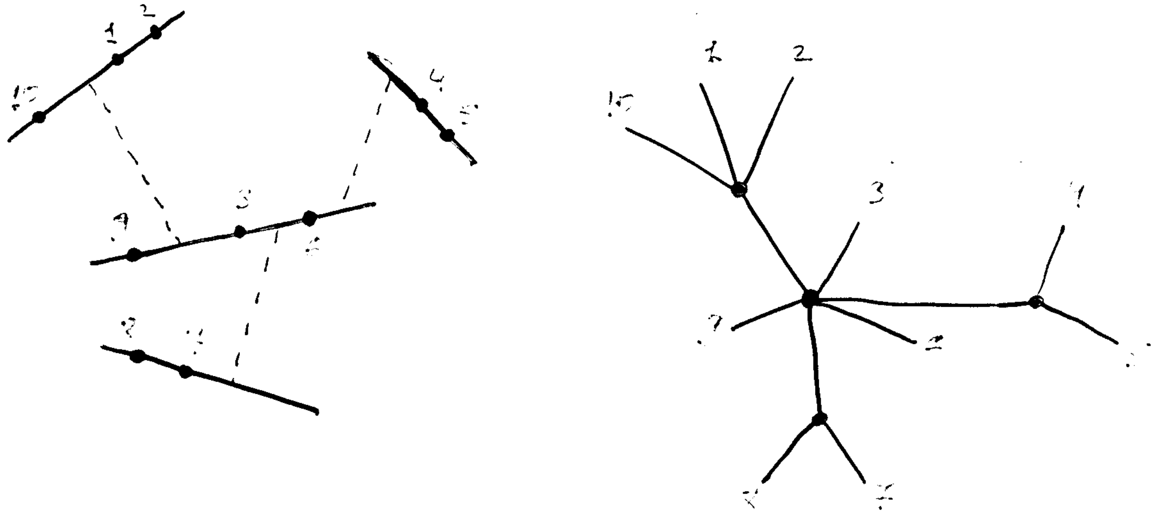
* **Surprise:**

$$\mathcal{A}_{YM}^{conn} = \mathcal{A}_{YM}$$

R. Roiban, M. Spradlin, A. Volovich

Completely Disconnected Instantons

Γ :



- $Q_i : \mathbb{CP}^1 \longrightarrow \mathbb{CP}^{3|4}$

  $$Q_i^{\mathcal{A}}(\sigma) = \sum_{k=0}^{1} \beta^{\mathcal{A}}_{k,i}\;\sigma^k$$

- moduli $\beta^{\mathcal{A}}_{k,i} \in \mathbb{C}^{8d|8d}$

- $GL(2,\mathbb{C})^d \implies 4d\,|\,8d$

$$\mathcal{M}^{\Gamma}_{lines} = \prod_{i=1}^{d} \mathcal{M}_{0,n_i}\left(\mathbb{CP}^{3|4}\right)$$

(Holomorphic) Measure:

i) $\mu_{lines} = \prod_{K,A,i} d\beta^{A}_{K,i}$

ii) $\omega_i := \omega(\sigma_1, \dots \sigma_{n_i})$

iii) $\Phi = \prod_i ev_i^* (\phi_i)$

iv) $D(x,y)$ (0,2)-form on $\mathbb{CP}^{3|4} \times \mathbb{CP}^{3|4}$
"holomorphic CS propagator"

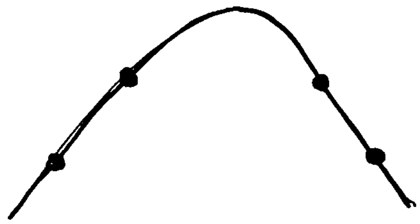
$$\boxed{\mathcal{A}^{disc}_{YM} = \sum_{\Gamma} \int_{\mathcal{M}^{\Gamma}_{lines}} \frac{\mu_{lines} \wedge \left(\prod_{i=1}^{d} \omega_i\right) \wedge \Phi \wedge D}{\left(GL(2,\mathbb{C})\right)^d}}$$

* Yet another surprise:

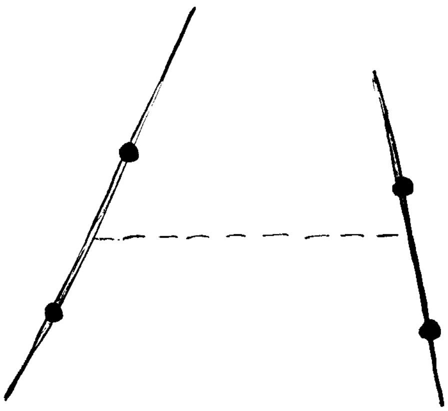
$$\mathcal{A}^{disc}_{YM} = \mathcal{A}_{YM}$$

F. Cachazo,
P. Svrcek,
E. Witten

Puzzle:

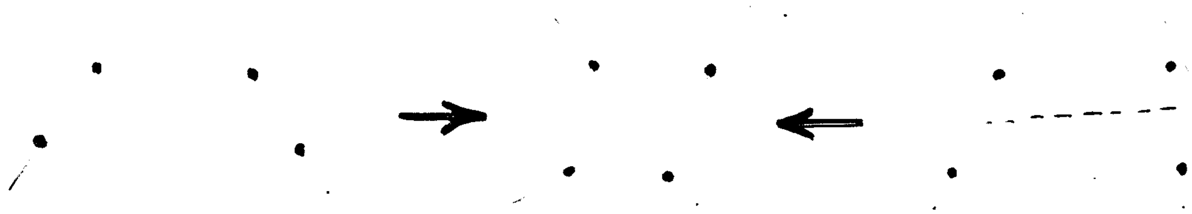


$$\mathcal{A}_{YM} = \int_{\overline{\mathcal{M}}_{0,n,d}} \omega_{conn}$$



$$\mathcal{A}_{YM} = \int_{\mathcal{M}_{lines}} \omega_{disc}$$

- both integrals have a simple pole along $\underline{\underline{\mathcal{M}_{int}}}$:

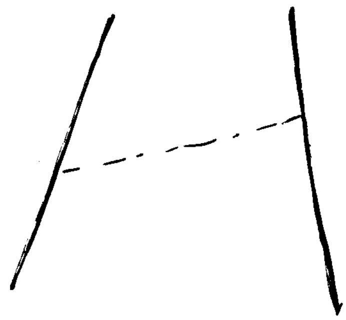


- same residue $\Rightarrow$

$$\int_{\overline{\mathcal{M}}_{0,n,d}} \omega_{conn} = \int_{\mathcal{M}_{lines}} \omega_{disc}$$

- ## Residue from ω_{disc}

$\mu^{\dot a} + x^{a\dot a}\lambda_a = 0$



$\mu^{\dot a} + x'^{a\dot a}\lambda_a =$

$$y^{a\dot a} := x'^{a\dot a} - x^{a\dot a} \quad \Rightarrow$$

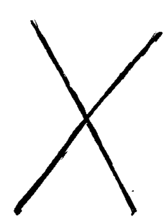
$$\omega_{disc} \propto \frac{d^4 y}{y_{a\dot a}\, y^{a\dot a}} = \frac{dy^{1\dot1} dy^{2\dot2} dy^{1\dot2} dy^{2\dot1}}{2\left(y^{1\dot1}y^{2\dot2} - y^{1\dot2}y^{2\dot1}\right)}$$

$$\Rightarrow \; res' = \frac{dy^{1\dot1}\, dy^{2\dot1} dy^{1\dot2}}{2y^{1\dot1}}$$

- ## Residue from ω_{conn}

$$\mu_d \sim \frac{d\varepsilon}{\varepsilon^3}$$

$\underline{\varepsilon = 0}$:



$$\omega(\sigma_1, \ldots, \sigma_n) \sim \varepsilon^2$$

$$\Rightarrow \quad \boxed{\omega_{conn} \sim \frac{d\varepsilon}{\varepsilon}}$$

Open Questions

- Choice of contours
- External wavefunctions
- Loops and higher genus
- Choice of prescriptions

⋮

- Dual String Theory?