# GW, GV, DT Invariants of CY 3folds

## I $DT^{\emptyset}$ invts (Thomas, MNOP)

$X$ nonsing. complex proj. 3 fold, $\beta \in H_2(X, \mathbb{Z})$, $n \in \mathbb{Z}$

$I_n(X,\beta)$ = moduli space (Hilb scheme) of ideals $\mathcal{I}_Z \subset \mathcal{O}_X$ defining a subscheme $Z \subset X$, $\mathcal{O}_Z = \mathcal{O}_X / \mathcal{I}_Z$)

① $[Z] = \beta$

② $\chi(\mathcal{O}_Z) = n$

Virtual dimension $D = \dim \mathrm{Ext}^1_0(\mathcal{I}_Z, \mathcal{I}_Z) - \dim \mathrm{Ext}^2_0(\mathcal{I}_Z, \mathcal{I}_Z) = c_1(X) \cdot \beta$

Virt. fund class $[I_n(X,\beta)]^{vir} \in H_{2D}(I_n(X,\beta), \mathbb{Z})$

If $\beta = 0$ then $D = 0$ $\forall n$.

$D_0^n := \deg [I_n(X,0)]^{vir} \in \mathbb{Z}$

$Z_0^{DT}(X) = \sum D_0^n q^n$

Conj (MNOP). $Z_0^{DT}(X) = M(-q)^{c_3 - c_1 c_2}$, where $M(q) = \prod (1-q^n)^{-n}$
$= (1 + q + 3q^2 + \dots)$

Proven if: $n=1$ (by construction), $n=2$ (MP) $X$ toric $\forall n$ (MNOP)

$n=3$ $X$ CY

For $\beta \neq 0$, get invariants by ~~[illegible]~~ imposing extra conditions MNOP using torus action (Bryan-Pandharipande; Elezi-K)

Can also achieve $D=0$ by $X = CY$.

$D^n_\beta := \deg [I_n(X,\beta)]^{vir} \in \mathbb{Z}$ . $Z^{DT}_\beta(X) = \sum D^n_\beta q^n$

$Z^{DT}(X) := \sum Z^{DT}_\beta(X)\, t^\beta$ $\qquad Z^{DT}(X)' = Z^{DT}(X) / Z^{DT}_0(X) =: \sum Z^{DT}_\beta(X)'\, t^\beta$

Recall GW invts. $N^g_\beta = \deg [\overline{M}_{g,0}(X,\beta)]^{vir} \in \mathbb{Q}$

$F'_g(X) = \sum_{\beta \neq 0} N^g_\beta t^\beta$ $\qquad F'(X) = \sum \lambda^{2g-2} F_g$ $\qquad Z^{GW}(X)' = \exp(F'(X))$

Conj. [MNOP] $Z^{DT}(X)' = Z^{GW}(X)'$ after change of vars $q = -e^{i\lambda}$

MNOP: If $X$ local toric, can define $Z^{DT}(X)'$ using any compactification
$Z^{DT}_\beta(X)'\,(q = -e^{i\lambda})$ = top. vertex prediction for $Z^{GW}(X)'$.

DT invts are often easy to compute in CY case: Suppose $I_n(X,\beta)$ is smooth. Then for $I \in I_n(X,\beta)$, $Ext^1_0(I,I) = T_{I_n(X,\beta),I}$. Since $Ext^2_0(I,I) \cong Ext^1_0(I,I)^*$, the obst. bundle is $T^* I_n(X,\beta)$, and, if $I_n(X,\beta) = \cup M_i$ conn comp.

Simple criterion: $D^n_\beta = \sum e(T^*M_i) = \sum (-1)^{\dim M_i} e(M_i)$.

<u>Prop</u>: $D^3_0$ = coeff $q^3$ in $M(-q)^{c_3}$

<u>Pf</u>. $X^{[3]} = I_3(X,0)$ smooth, blowup of $Sym^3 X$ along diagonals $\Delta \supset \Delta_3$ small ~~small diagonal~~

$$X^{[3]} = (Sym^3 X - \Delta) \cup (\mathbb{P}^2 \text{ bundle over } \Delta - \Delta_3) \cup (\mathbb{P}^5 \text{ bundle over } \Delta_3)$$

Put $e = e(X)$

$$e(X^{[3]}) = \tfrac{1}{6} e(X^3 - \Delta) + 3\, e(\Delta - \Delta_3) + 6\, e(\Delta_3)$$
$$= \tfrac{1}{6}(e^3 - 3e^2 + 2e) + 3(e^2 - e) + 6e ;$$

compute $(1 - q + 3q^2 - 6q^3 + O(q^4))^e$ $\square$

II GV invariants.

From $M[X]$ have GV invts $n^g_\beta \in \mathbb{Z}$ (conj, only proven $\in \mathbb{Q}$) s

$$F'(X) = \sum_{m,g} n^g_\beta \frac{1}{m} \left(2 \sin \frac{m\lambda}{2}\right)^{2g-2} t^\beta .$$

Ex: local $\mathbb{P}^1$ $\quad n^0_{[\mathbb{P}^1]} = 1$

generic ellipt. $E$ $\quad n^1_{k[E]} = 1 \quad \forall\ k \geq 1$

Let $M$ be a connected component of $I_{1-g}(X, \beta)$ consisting generically of ideals of smooth curves $C \subset X$ of genus $g$. Let $\mathcal{C} \subset M \times X$ be the univ. curve, $\mathcal{C}^{[n]}$ = rel. Hilb scheme of $n$ points

$$\mathcal{C}^{[n]} = \{ C, p_1, \ldots, p_n \in C_i : C_i \in M \}. \quad \text{Put } \mathcal{C}^{[1]} = \mathcal{C},\ \mathcal{C}^{[0]} = M$$

Assume $\mathcal{C}^{[n]}$ smooth, $n \leq \delta$. Then under additional mild hypotheses

Conj: (K Klemm Vafa) $\quad (-1)^{\dim M + \delta}\, n^{g-\delta}_{\beta} = e(\mathcal{C}^{[\delta]})$

$$+ (2g - 2\delta)\, e(\mathcal{C}^{[\delta-1]}) + \sum_{i=2}^{\delta} \frac{1}{i!} (2g - 2\delta + 2i - 2)(2g - 2\delta + i - 3)(2g - 2\delta + i - 4) \cdots (2g - 2\delta - 1)\, e(\mathcal{C}^{[\delta - i]})$$

Prop a) Under same hypotheses, & $\delta \leq 3$, contribution of $M$ to $Z^{DT}_\beta(X)'$ is

$$\sum_{n=0}^{\delta} (-1)^{\dim \mathcal{C}^{[n]}} e(\mathcal{C}^{[n]})\, q^{n+1-g} + O(q^{\delta+2-g})$$

b) The KKV conj implies a).

Ex: deg 3, local $\mathbb{P}^2$ $g=1$ $\quad I_0(\mathbb{P}^2, 3) = \mathbb{P}^9$ $\quad$ $\mathbb{P}^8 \to \mathcal{C} \subset \mathbb{P}^2 \times \mathbb{P}^9$, $\mathcal{C} \to \mathbb{P}^2$ $\quad e(\mathcal{C}) = 3 \cdot 9 = 27$

$n^0_3 = e(\mathcal{C}^{[1]}) \cancel{+} (0)\, e(\mathcal{C}^{[0]}) = 27$

$n^1_3 = -e(\mathcal{C}^{[0]}) = -10$ $\quad$ $D^0_3 = -10$ $\quad$ ~~$I_1(\mathbb{P}^2, 3)$~~

Contribution of genus $g$ GV inv't to $Z'_{DT}$:

(iii)

$$Z'_{DT} = \exp\left( \sum \frac{1}{m} \left(2\sin \frac{m\lambda}{2}\right)^{2g-2} t^{\beta} \right)$$

$(g>0)$

$$= \prod_{k=0}^{2g-2} \left(1 - e^{i(g-1-k)\lambda} t^{\beta}\right)^{(-1)^{k+g} \binom{2g-2}{k}}$$

$$= \prod_{k=0}^{2g-2} \left( 1 + (-1)^{g-k} q^{g-1-k} t^{\beta} \right)^{(-1)^{k+g} \binom{2g-2}{k}}$$

Similar expression for $g=0$

Note: leading term $q^{1-g} t^{\beta}$, $\chi(\mathcal{O}) = 1-g$.

Ex: $g=1$ $\quad Z'_{DT} = \prod_m (1 - t^{m\beta})^{-1} = \sum p(k) t^{k\beta}$

Locally, $X = L \oplus L^{-1} \to C$ $\quad$ deg $L = 0$, $\quad$ $L, L^{-1}$ only deg 0 subbundles of $X$.

Let $x, y$ be fiber coords on $L, L^{-1}$, 0 section $C$ is $x=y=0$

Given a partition of $n$ get an ideal in $(x,y)$ defining a subscheme in $nC$

e.g. $x^2, xy, y^2$

Now, suppose $X$ has $\ell$ ell. curve $E_1, \ldots, E_\ell$ in class $\beta$; should have $Z_\beta^{DT}(X)' = \ell$.

$I_1(X,\beta) = \bigcup Bl_{E_i} X \qquad e(Bl_{E_i} X) = e(X) + e, \quad D_\beta^1 = e$

Sim $I_n(X,\beta) \qquad D_\beta^n = e(X^{[n]}) \quad (n \leq 3)$

$\Rightarrow Z_\beta^{DT}(X) \equiv \ell M(-q)^e \quad$ mod $q^4$

$\Rightarrow Z_\beta^{DT}(X)' \equiv \ell \quad$ mod $q^4$.

Pf of KKV implicat. $\quad I_{1-g}(X,\beta) = \mathcal{M} \Rightarrow D_\beta^{1-g} = (-1)^{\dim \mathcal{M}} e(\mathcal{M})$

$I_{2-g}(X,\beta) = Bl_{\mathcal{C}}(\mathcal{M} \times X) \Rightarrow D_\beta^{2-g} = (-1)^{\dim \mathcal{M}+1} \left(e(\mathcal{M}) e(X) + e(\mathcal{C})\right)$

$\Rightarrow Z_\beta^{DT} = (-1)^{\dim \mathcal{M}} e(\mathcal{M}) q^{1-g} + (-1)^{\dim \mathcal{M}+1} \left(e(\mathcal{M}) e(X) + e(\mathcal{C})\right) q^{2-g} + \ldots$

$\Rightarrow Z_\beta^{DT'} = Z_\beta^{DT}/Z_{0,\beta} = (-1)^{\dim \mathcal{M}} e(\mathcal{M}) q^{1-g} + (-1)^{\dim \mathcal{M}+1} e(\mathcal{C}) q^{2-g} + \ldots$

$q^{3-g}$, sim

KKV implicit in a simple calc

□