Some Correlation Function Computations in Heterotic (0,2) Models

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Work In Progress w/ S. Katz

Today I'll describe some calculations of heterotic correlation f'ns that are modelled on curve-counting in the A model TFT.

Motivation: (0,2) mirror symmetry

Ordinary mirror symmetry: $X_1 \longleftrightarrow X_2$, X_1, X_2 CY

(0,2) mirror symmetry: $(X_1, \mathcal{E}_1) \longleftrightarrow (X_2, \mathcal{E}_2)$

where $\mathcal{E}_1, \mathcal{E}_2$ are bundles on X_1, X_2

(0,2) mirror symmetry is poorly understood at present.

Recently, Adams-Basu-Sethi studied (0,2) mirrors.
They applied old work of Morrison-Plesser,
more recently explained by Hori-Vafa,
to (0,2) GLSM's to make some predictions for
(0,2) mirrors in some relatively simple cases.

They also made some predictions for product structures
in heterotic chiral rings,
which we are trying to verify,
and is the subject of today's talk.

Outline

- review A model TFT, $\frac{1}{2}$-twisted (0,2) TFT
- review correlation f'n computations in A model, describe analogue for (0,2) models
  - formal structure similar; (0,2) generalizes A model
  - compactification issues; not only $\mathcal{M}$, but sheaves on $\mathcal{M}$
- Apply GLSM's
  - not only naturally compactify $\mathcal{M}$, but also naturally extend the sheaves
- Adams-Basu-Sethi prediction

Recall the A-model:

$$g_{i\bar{j}}\bar{\partial}\phi^i\partial\phi^{\bar{j}} + i g_{i\bar{j}}\psi_-^{\bar{j}} D_z \psi_-^i + i g_{i\bar{j}}\psi_+^{\bar{j}} D_{\bar{z}}\psi_+^i + R_{i\bar{j}k\bar{l}}\psi_+^i\psi_+^{\bar{j}}\psi_-^k\psi_-^{\bar{l}}$$

$\psi_-^i \, (\equiv \chi^i) \in \Gamma(\phi^* T^{1,0}X)$ $\qquad$ $\psi_+^i \, (\equiv \psi_z^i) \in \Gamma(K\otimes \phi^* T^{1,0}X)$

$\psi_-^{\bar{\imath}} \, (\equiv \psi_{\bar{z}}^{\bar{\imath}}) \in \Gamma(K\otimes\phi^* T^{0,1}X)$ $\qquad$ $\psi_+^{\bar{\imath}} \, (\equiv \chi^{\bar{\imath}}) \in \Gamma(\phi^* T^{0,1}X)$

States: $\mathcal{O} \sim b_{i_1\cdots i_p,\bar{\imath}_1\cdots\bar{\imath}_q}(\phi)\,\chi^{\bar{\imath}_1}\cdots\chi^{\bar{\imath}_q}\chi^{i_1}\cdots\chi^{i_p}$

$$\longleftrightarrow H^{p,q}(X)$$

Half-twisted (0,2) model:

$$g_{i\bar{j}}\bar{\partial}\phi^i\partial\phi^{\bar{j}} + i h_{a\bar{b}}\lambda_-^{\bar{b}} D_z \lambda_-^a + i g_{i\bar{j}}\psi_+^{\bar{j}} D_{\bar{z}}\psi_+^i + F_{i\bar{j}a\bar{b}}\psi_+^i\psi_+^{\bar{j}}\lambda_-^a\lambda_-^{\bar{b}}$$

$\lambda_-^a \in \Gamma(\phi^*\mathcal{E})$ $\qquad$ $\psi_+^i \in \Gamma(K\otimes\phi^* T^{1,0}X)$

$\lambda_-^{\bar{a}} \in \Gamma(K\otimes\phi^*\mathcal{E})$ $\qquad$ $\psi_+^{\bar{\imath}} \in \Gamma(\phi^* T^{0,1}X)$

RR states: $\mathcal{O} \sim b_{\bar{\imath}_1\cdots\bar{\imath}_n,a_1\cdots a_r}(\phi)\,\psi_+^{\bar{\imath}_1}\cdots\psi_+^{\bar{\imath}_n}\lambda_-^{a_1}\cdots\lambda_-^{a_r}$

$$\longleftrightarrow H^n(X, \Lambda^r\mathcal{E}^\vee)$$

When $\mathcal{E} = TX$, reduces to the A model above.

Symmetry properties of states:

A model:

$$H^{p,q}(X) \simeq H^{n-p,n-q}(X) \quad \text{for compact } n\text{-dim'l } X$$

(0,2) model:

$$H^{q}(X, \Lambda^{p}\mathcal{E}^{\vee}) \simeq H^{n-q}(X, (\Lambda^{r-p}\mathcal{E}^{\vee}) \otimes (\Lambda^{top}\mathcal{E} \otimes K_X))$$

for compact n-dim'l X
rank r $\mathcal{E}$

We'll assume $\Lambda^{top}\mathcal{E}^{\vee} \simeq K_X$,

in add'n to anomaly cancellation $ch_2(\mathcal{E}) = ch_2(TX)$

- recovers symmetry property $H^{q}(X, \Lambda^{p}\mathcal{E}^{\vee}) \simeq H^{n-q}(X, \Lambda^{r-p}\mathcal{E}^{\vee})$
- essential for correlation functions
- in CY compactification, guarantees a left-moving $U(1)$ that is essential for spacetime gauge symmetry

Anomaly cancellation

We just outlined one property that the bundles $\mathcal{E}$ will be assumed to possess, namely $\Lambda^{top}\mathcal{E}^\vee \cong K_X$.

We shall also assume $ch_2(\mathcal{E}) = ch_2(TX)$.

In a CY compactification, this is "anomaly cancellation" condition arising from spacetime statement

$$dH = \mathrm{tr}\, F \wedge F - \mathrm{tr}\, R \wedge R$$

This condition also manifests itself in the worldsheet theory, and can be derived (as we'll see later) for massive 2D QFT's w/ non-CY targets.

Summary of conditions on $\mathcal{E}$:

$$\Lambda^{top}\mathcal{E}^\vee \cong K_X$$

$$ch_2(\mathcal{E}) = ch_2(TX)$$

Why work with the A model, or this "half-twisted" theory?
Why not work directly in physical untwisted theories?

Reason: the twisted theories give same answer, with less work.

Consider 3-point functions.

A model:

It's an old story that

$$\langle \psi \psi \phi \rangle_{phys, II} = \langle \psi \psi \psi \rangle_A$$

b/c the spectral flow operator encoding $\phi \leftrightarrow \psi$ is equivalent to twisting the theory.

After all, twist by adding $\sim \int \frac{1}{2} \omega \bar{\psi} \psi = \int \frac{1}{2} \omega J$ to action
If bosonize $J \sim \partial \phi$, then the term $\sim \int R \phi$
By concentrating curvature at points, so $R \sim \delta^2(z - z_0)$
we see that twisting $\sim$ inserting $\exp(\phi)$ $\sim$ spectral flow.

3-pt f'ns, cont'd

(0,2) model

No longer have left-moving $N=2$ susy, but do have a left-moving $U(1)$ that becomes $U(1)_R$ on (2,2) locus, & is crucial for gauge properties.

Ex $\mathcal{E}$ is rank 3, breaking E_8 to E_6.
E_6 is built from $SO(10) \times \underbrace{U(1)}_{\Sigma}$

$$\overline{27} = 10_{-1} \oplus \overline{16}_{1/2} \oplus 1_2$$

so $\overline{27}^3$ calculated as $\langle \psi_{\overline{16}} \psi_{\overline{16}} \phi_{10} \rangle_{phys, het}$

For same reasons as for the A model,

$$\langle \psi_{\overline{16}} \psi_{\overline{16}} \phi_{10} \rangle_{\substack{phys, \\ het}} = \langle \psi \psi \psi \rangle_{\frac{1}{2}\text{ twisted}}$$

Classical correlation functions (no worldsheet instantons)

A model:

For X compact, n-dim'l, have n χ^i zero modes, n $\chi^{\bar{\imath}}$ zero modes, bosonic zero modes $\sim X$,

$$\therefore \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X H^{p_1,q_1}(X) \wedge \cdots \wedge H^{p_m,q_m}(X)$$

Selection rule from left-, rt-moving $U(1)$'s : $\sum p_i = \sum q_i = n$

$$\therefore \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X (\text{top-form})$$

(0,2) model:

n $\psi_+^{\bar{\imath}}$ zero modes, r λ^a zero modes

$$\therefore \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X H^{q_1}(X, \Lambda^{p_1} \mathcal{E}^\vee) \wedge \cdots \wedge H^{q_m}(X, \Lambda^{p_m} \mathcal{E}^\vee)$$

Selection rule from left-, rt-moving $U(1)$'s : $\sum q_i = n$, $\sum p_i = r$

$$\therefore \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_X H^{top}(X, \Lambda^{top} \mathcal{E}^\vee)$$

When $\Lambda^{top} \mathcal{E}^\vee \cong K_X$, then this is a top-form.

Next: worldsheet instantons

A model:

TFT localizes on holomorphic maps,
so moduli space of bosonic zero modes
= moduli space of worldsheet instantons $\mathcal{M}$

We'll assume $\mathcal{M}$ is smooth, & review compactification later.

(0,2) model:

In add'n to $\mathcal{M}$,
the bundle $\mathcal{E}$ on X induces sheaf $\mathcal{F}$ on $\mathcal{M}$.

$$\mathcal{F} \equiv R^0 \pi_* \alpha^* \mathcal{E} \quad \text{where} \quad \begin{array}{l} \alpha : \Sigma \times \mathcal{M} \to X \\ \pi : \Sigma \times \mathcal{M} \to \mathcal{M} \end{array}$$

On (2,2) locus, where $\mathcal{E} = TX$, have $\mathcal{F} = T\mathcal{M}$ [fixed & str'] on worldsheet]

When no excess zero modes $(R^1 \pi_* \alpha^* \mathcal{E} = 0 = R^1 \pi_* \alpha^* TX)$,

$$\left.\begin{array}{l} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ ch_2(\mathcal{E}) = ch_2(TX) \end{array}\right\} \xRightarrow{GRR} \Lambda^{top} \mathcal{F}^\vee \cong K_{\mathcal{M}}$$

A model (no excess zero modes – no $\psi_z^i, \psi_{\bar z}^{\bar\imath}$)

$$\mathcal{O}_i \sim H^{p_i, q_i}(X) \longrightarrow H^{p_i, q_i}(\mathcal{M})$$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{p_1, q_1}(\mathcal{M}) \wedge \cdots \wedge H^{p_m, q_m}(\mathcal{M})$$

Selection rules $\Rightarrow$ $\sum p_i = \sum q_i = \dim \mathcal{M}$

$$\therefore \ \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{top, top}(\mathcal{M})$$

(0, 2) model (no excess zero modes – no $\psi_+^i, \lambda^{\bar a}$)

$$\mathcal{O}_i \sim H^{q_i}(X, \Lambda^{p_i} \mathcal{E}^\vee) \longrightarrow H^{q_i}(\mathcal{M}, \Lambda^{p_i} \mathcal{Z}^\vee)$$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{q_1}(\mathcal{M}, \Lambda^{p_1} \mathcal{Z}^\vee) \wedge \cdots \wedge H^{q_m}(\mathcal{M}, \Lambda^{p_m} \mathcal{Z}^\vee)$$

Selection rules $\Rightarrow$ $\sum q_i = \dim \mathcal{M}$, $\sum p_i = \operatorname{rank} \mathcal{Z}$

$$\therefore \ \langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{top}(\mathcal{M}, \Lambda^{top} \mathcal{Z}^\vee)$$

& recall $\Lambda^{top} \mathcal{Z}^\vee \cong K_{\mathcal{M}}$ (from anomaly cancellation, GRR)

$\therefore$ $H^{top}(\mathcal{M}, \Lambda^{top} \mathcal{Z}^\vee) \sim$ top form

How is cohomology on X mapped to cohomology on $\mathcal{M}$?

<u>A model</u>

Each element of $H^{p,q}(X)$ + point p on worldsheet define element of $H^{p,q}(\mathcal{M})$ by,

pullback along $\alpha|_{p \times \mathcal{M}}$, where $\alpha: \Sigma \times \mathcal{M} \to X$

<u>(0,2) model</u>

Each element of $H^q(X, \Lambda^p \mathcal{E}^\vee)$ + point p on worldsheet define element of $H^q(\mathcal{M}, \Lambda^p \mathcal{F}^\vee)$:

- first pullback along $\alpha|_{p \times \mathcal{M}}$ to get $\in H^q(\mathcal{M}, \Lambda^p (\alpha^* \mathcal{E})^\vee|_{p \times \mathcal{M}})$
- next use map

$$\mathcal{F} \; (\equiv \pi_* \alpha^* \mathcal{E}) \longrightarrow \alpha^* \mathcal{E}|_{p \times \mathcal{M}}$$

to define map

$$\Lambda^p (\alpha^* \mathcal{E})^\vee|_{p \times \mathcal{M}} \longrightarrow \Lambda^p \mathcal{F}^\vee$$

When $\mathcal{E} = TX$, this reduces to the A model map.

How to handle excess zero modes?

A model

Use 4-fermi term $\int_\Sigma R_{i\bar{j}k\bar{l}} \, \chi^i \chi^{\bar{j}} \psi^k \psi^{\bar{l}}$

For each $\mathbb{C}$ pair of ψ zero modes, bring down one copy of 4-fermi term above.

Result:

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{\sum p_i, \sum q_i}(\mathcal{M}) \wedge c_{top}(Obs)$$

where Obs = sheaf over $\mathcal{M}$ defined by ψ zero modes
$= R^1 \pi_* \alpha^* TX$
= "obstruction sheaf"

Selection rules say

$$\sum p_i = \sum q_i = \# \chi - \# \psi \text{ zero modes}$$

$\# \psi$ zero modes = rank Obs
$\# \chi$ " " = dim $\mathcal{M}$

$\therefore \sum p_i + (\text{rank } Obs) = \sum q_i + (\text{rank } Obs) = \dim \mathcal{M}$

$\therefore$ integrand is a top-form

Excess zero modes, cont'd

(0, 2) model

Assume rank $R^1\pi_* \alpha^* \mathcal{E}$ = rank $R^1 \pi_* \alpha^* TX \equiv n$

Use 4-fermi term $\int_\Sigma F_{i\bar{j}a\bar{b}} \, \psi_+^i \psi_+^{\bar{j}} \lambda^a \lambda^{\bar{b}}$

$$\psi^{\bar{j}} \sim T\mathcal{M} = R^0\pi_* \alpha^* TX \qquad \lambda^a \sim \mathcal{F} = R^0 \pi_* \alpha^* \mathcal{E}$$
$$\psi^{i} \sim Obs = R^1\pi_* \alpha^* TX \qquad \lambda^{\bar{b}} \sim \mathcal{F}_1 \equiv R^1 \pi_* \alpha^* \mathcal{E}$$

Each 4-fermi $\sim H^1(\mathcal{M}, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes (Obs)^\vee)$

$$\langle \mathcal{O}_1 \cdots \mathcal{O}_m \rangle \sim \int_{\mathcal{M}} H^{\sum q_i}(\mathcal{M}, \Lambda^{\sum p_i} \mathcal{F}^\vee) \wedge H^n(\mathcal{M}, \Lambda^n \mathcal{F}^\vee \otimes \Lambda^n \mathcal{F}_1 \otimes \Lambda^n (Obs)^\vee)$$

Selection rules:
$$\sum q_i + n = \dim \mathcal{M}$$
$$\sum p_i + n = \text{rank } \mathcal{F}$$
(by assumption, rank $\mathcal{F}_1$ = rank $Obs = n$

$$\left. \begin{array}{l} \Lambda^{top} \mathcal{E}^\vee \cong K_X \\ ch_2(\mathcal{E}) = ch_2(TX) \end{array} \right\} \xrightarrow{GRR} \Lambda^{top} \mathcal{F} \otimes \Lambda^{top} \mathcal{F}_1^\vee \cong \Lambda^{top} T\mathcal{M} \otimes \Lambda^{top} (Obs)^\vee$$

or, more simply,
$$\Lambda^{top} \mathcal{F}^\vee \otimes \Lambda^{top} \mathcal{F}_1 \otimes \Lambda^{top} (Obs)^\vee \cong K_{\mathcal{M}}$$

$\therefore$ once again, integrand is a top-form!

We've just presented an ansatz for interpreting 4-fermi terms in (0,2) theories, & observed that GRR $\Rightarrow$ integrand is a top-form, as needed.

But why does it reduce to (2,2) case when $\mathcal{E} = TX$?

Atiyah classes

Consider the curvature of a connection on a hol' bundle $\mathcal{E}$ on X:

$$F_{\underbrace{i\bar{\jmath}}_{TX}\,\underbrace{a\bar{b}}_{\mathrm{End}\,\mathcal{E}}}$$

Bianchi: $\bar{\partial} F = 0 \quad \therefore [F] \in H^1(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E})$

Since $ch_r(\mathcal{E}) \propto \mathrm{tr}\, \underbrace{F \wedge \cdots \wedge F}_{r \text{ times}}$,

the Chern classes of $\mathcal{E}$ are encoded in

$$H^1(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}) \wedge \cdots \wedge H^1(X, \Omega^1_X \otimes \mathcal{E}^\vee \otimes \mathcal{E}) = H^r(X, \Omega^r_X \otimes \Lambda^r \mathcal{E}^\vee \otimes \Lambda^r \mathcal{E})$$

Let's specialize our (0,2) discussion to $\mathcal{E} = TX$, so $\mathcal{Z} = T\mathcal{M}$.

Each (0,2) 4-fermi term generates a factor of

$$H^1(\mathcal{M}, \mathcal{Z}^\vee \otimes \mathcal{Z}_1 \otimes (\text{Obs})^\vee) \underset{\mathcal{E}=TX}{=\!=} H^1(\mathcal{M}, \Omega^1_{\mathcal{M}} \otimes (\text{Obs})^\vee \otimes \text{Obs})$$

$\rightarrow$ same sheaf cohomology group that contains the Atiyah class of Obs

Bringing down (n = rank Obs) factors generates

$$H^n(\mathcal{M}, \Omega^n_{\mathcal{M}} \otimes \Lambda^{top}(\text{Obs})^\vee \otimes \Lambda^{top}\text{Obs})$$

which contains $c_{top}(\text{Obs})$.

$\therefore$ our (0,2) ansatz generalizes (2,2) obstruction bdle story

Next: compactifications of $\mathcal{M}$

Compactifications of moduli spaces

In order to make sense of expressions such as

$$\int_{\mathcal{M}} (\text{top-form})$$

need $\mathcal{M}$ to be compact.

Problem: spaces of honest holomorphic maps not compact

Ex Degree 1 maps $\mathbb{P}^1 \to \mathbb{P}^1$
$= $ group manifold of $SL(2, \mathbb{C})$

Furthermore, in the $(0,2)$ case,
need to extend $\mathcal{F}_1$, $\mathcal{F}_1$ over the compactification
in a way consistent w/ symmetries.

How to compactify?
One way (Morrison-Plesser) uses gauged linear sigma models.
We'll follow their lead,
and will see that GLSM's also naturally describe
how to extend $\mathcal{F}_1$, $\mathcal{F}_1$.

Next: review of GLSM's

Gauged linear sigma models:

(2,2) case:

A chiral superfield contains

ϕ ($\mathbb{C}$ boson)
ψ_+, ψ_- ($\mathbb{C}$ fermions)
F (auxiliary field)

A GLSM describes $\mathbb{P}^{N-1}$ as,
N chiral superfields each of charge 1 w.r.t. gauged $U(1)$

D-terms: $\sum |\phi_i|^2 = r \Rightarrow \phi$'s span S^{2N-1}
Gauge-invariants: $S^{2N-1}/U(1) = \mathbb{P}^{N-1}$

Can use GLSM's to describe more general toric varieties
→ look like, some chiral superfields + gauged $U(1)$'s

Can describe CY's by adding superpotential;
zero locus of bosonic potential = CY.

- massive 2D QFT's, not CFT's
- linear kinetic terms make analysis of some aspects of QFT easier than in a NLSM

Today I'll only consider (mostly massive) theories w/ toric targets.

(0,2) GLSM's

(Distler–Kachru)

(0,2) chiral superfield Φ	(0,2) fermi superfield Λ
ϕ ($\mathbb{C}$ boson)	ψ_- ($\mathbb{C}$ fermion)
ψ_+ ($\mathbb{C}$ fermion)	F (auxiliary field)

Together, form a (2,2) chiral multiplet.

The fermi superfields have an important quirk:

Although $\overline{D}_+ \Phi = 0$ for Φ chiral,
can permit $\overline{D}_+ \Lambda = E$ for nonzero E obeying $\overline{D}_+ E = 0$.
This constrains the superpotential.
Details soon....

Can describe a toric variety target
as a collection of (0,2) chiral superfields
with some gauged $U(1)$'s.

Since fermi multiplets are ~ left-moving,
might correctly guess they define bundles.

Next: description of bundles in GLSM's

Anomaly cancellation

Let $\bar{n}_a$ denote charges of left-moving fermions,
" $\bar{q}_i$ " " " right- " ".

Anomaly cancellation implies

$$\sum_a n_a^t n_a^s = \sum_i q_i^t q_i^s \quad \text{for each } s,t$$

This implies, but is slightly stronger than, $ch_2(\mathcal{E}) = ch_2(TX)$

We'll also assume $\sum n_a^t = \sum q_i^t$ for each t,

which implies, but is slightly stronger than, $c_1(\mathcal{E}) = c_1(TX)$.

Bundles on the toric variety are described w/ aid of (0, 2) fermi superfields.

Ex Reducible case: $\mathcal{E} = \bigoplus_a \mathcal{O}(\vec{n}_a)$

In GLSM have fermi superfields Λ^a w/ charges $\vec{n}_a$ under some $U(1)$'s.

Ex Kernel: $0 \to \mathcal{E} \to \bigoplus_a \mathcal{O}(\vec{n}_a) \xrightarrow{F_a^i} \bigoplus_i \mathcal{O}(\vec{m}_i) \to 0$

Have fermi superfields Λ^a as above, plus chiral superfields p_i of charges $\vec{m}_i$ plus superpotential term $p_i F_a^i(\phi) \Lambda^a$

Resulting Yukawa couplings $\psi_{p+i} F_a^i(\phi) \lambda^a$ give mass to any λ not in $\ker F$, hence, $\mathcal{E} = \ker F$

Ex Cokernel: $0 \to \mathcal{O}^k \xrightarrow{E_a^\lambda} \bigoplus_a \mathcal{O}(\vec{n}_a) \to \mathcal{E} \to 0$

Have fermi superfields Λ^a w/ charges $\vec{n}_a$ as above, plus k neutral chiral superfields Σ_λ, where $\overline{D}_+ \Lambda_a = \Sigma_\lambda E_a^\lambda(\phi)$

Ex Monad: $0 \to \mathcal{O}^k \xrightarrow{E_a^\lambda} \bigoplus_a \mathcal{O}(\vec{n}_a) \xrightarrow{F_a^i} \bigoplus_i \mathcal{O}(\vec{m}_i) \to 0$; $\mathcal{E} \equiv \dfrac{\ker F_a^i}{\operatorname{im} E_a^\lambda}$

Have Σ_λ, Λ^a, p_i as above, w/ superpotential and susy trans!

Linear sigma model compactifications

Basic idea: • expand fields in a basis of zero modes
• coefficients are homogeneous coord's on $\mathcal{M}$

Ex $\mathbb{P}^{N-1}$

N chiral superfields, $x_1, \cdots x_N$
1 gauged $U(1)$, each x_i has charge 1

The gauge instantons of the GLSM
become the worldsheet instantons of the NLSM.

Moduli space of degree d maps in this example:

$$x_i \in \Gamma(\mathcal{O}(1\cdot d))$$
$$x_i = x_{i0} u^d + x_{i1} u^{d-1} v + \cdots + x_{id} v^d$$

where u, v are homogeneous coordinates on worldsheet ($\mathbb{P}^1$)
The (x_{ij}) are homogeneous coord's on $\mathcal{M}$
Omit point where all $x_i \equiv 0$
The (x_{ij}) have same $U(1)$ charges as x_i for each x_i

$$\therefore \mathcal{M} = \mathbb{P}^{N(d+1)-1}$$

More generally, given x_i of charges $\overline{q}_i$,
a moduli space of maps of degree $\overline{d}$ determined by
$x_i \in \Gamma(\mathcal{O}(\overline{q}_i \cdot \overline{d}))$ as above.

The same ideas allow us to induce bundles on LSM moduli spaces.

Just as worldsheet fields define line bundles on target, expand in zero modes, and coefficients define line bundles on $\mathcal{M}$.

First case: completely reducible bundles

S'pose left-moving fermions are completely free (mod action of the gauge group), so

$$\mathcal{E} = \bigoplus_a \mathcal{O}(\vec{n}_a)$$

Expand each fermion in zero modes, take coeff's to define line bundles on $\mathcal{M}$.

Here, λ_-^a of charge $\vec{n}_a$.
Expand $\lambda_-^a = \lambda_-^{a0} u^{\vec{n}_a \cdot \vec{d}+1} + \lambda_-^{a1} u^{\vec{n}_a \cdot \vec{d}} v + \cdots$

Each $\lambda_-^{ai} \sim \mathcal{O}(\vec{n}_a)$ on $\mathcal{M}$

$$\therefore \mathcal{F} = \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$

Sim'ly,

$$\mathcal{F}_1 = \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$

S'pose $\mathcal{E}$ described as cokernel:

$$0 \to \mathcal{O}^{\oplus m} \longrightarrow \bigoplus_a \mathcal{O}(\vec{n}_a) \to \mathcal{E} \to 0$$

$$\Lambda^a$$

In add'n to fermi superfields for the $\mathcal{O}(\vec{n}_a)$, recall have chiral superfields $\hat{\Sigma}_\lambda$ for the $\mathcal{O}$'s. As before, expand fields in basis of zero modes, and interpret coefficients as line bundles on $\mathfrak{M}$.

$$0 \longrightarrow \bigoplus_1^m H^0(\mathbb{P}^1, \mathcal{O}(\vec{0}\cdot\vec{d})) \otimes_{\mathbb{C}} \mathcal{O} \longrightarrow \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a\cdot\vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a) \to \mathcal{Z}$$

$$\to \bigoplus_1^m H^1(\mathbb{P}^1, \mathcal{O}(\vec{0}\cdot\vec{d})) \otimes_{\mathbb{C}} \mathcal{O} \longrightarrow \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a\cdot\vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a) \to \mathcal{Z}_1 \to 0$$

Since $H^1(\mathbb{P}^1, \mathcal{O}) = 0$, this simplifies to

$$0 \to \mathcal{O}^{\oplus m} \longrightarrow \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a\cdot\vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a) \to \mathcal{Z} \to 0$$

$$\mathcal{Z}_1 \cong \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a\cdot\vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$

First check: (2,2) locus

The tangent bundle of a (cpt, smooth) toric variety X can be expressed in the form

$$0 \to \mathcal{O}^{\oplus k} \to \bigoplus_i \mathcal{O}(\vec{q}_i) \to TX \to 0$$

where the $\vec{q}_i$ are charges of the chiral superfields.

Applying previous ansatz,

$$0 \to \mathcal{O}^{\oplus k} \to \bigoplus_i H^0(\mathbb{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{q}_i) \to \mathcal{F} \to 0$$

$$\mathcal{F}_1 \simeq \bigoplus_i H^1(\mathbb{P}^1, \mathcal{O}(\vec{q}_i \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{q}_i)$$

but this $\mathcal{F}$ is automatically $T\mathcal{M}$ for $\mathcal{M}$ = LSM moduli space, exactly as desired.

Also, $\mathcal{F}_1$ = obstruction bundle.

$$\left[\text{Check: } c_{top}(\mathcal{F}_1) = \prod_{\vec{n}_a \cdot \vec{d} < 0} c_1(\mathcal{O}(\vec{n}_a))^{-\vec{n}_a \cdot \vec{d} - 1} \checkmark \right]$$

Next, check that if $\mathcal{E}$ satisfies anomaly cancellation, then $\mathcal{Z}, \mathcal{Z}_1$ have desired properties.

Recall
$$0 \to \mathcal{O}^{\oplus n} \longrightarrow \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a) \to \mathcal{Z} \to 0$$

$$\mathcal{Z}_1 \simeq \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$

Calculate
$$c_1(\mathcal{Z}) - c_1(\mathcal{Z}_1) = \left[\sum_{\vec{n}_a \cdot \vec{d} \geq 0} (\vec{n}_a \cdot \vec{d} + 1)\, n_a^t J_t \right] - \left[\sum_{\vec{n}_a \cdot \vec{d} < 0} (-\vec{n}_a \cdot \vec{d} - 1)\, n_a^t J_t \right]$$
$$= \left[\sum_a n_a^t J_t \right] + \left[\sum_a (\vec{n}_a \cdot \vec{d})\, n_a^t J_t \right]$$

Tangent bundle: $\quad 0 \to \mathcal{O}^{\oplus k} \to \bigoplus_i \mathcal{O}(\vec{q}_i) \to TX \to 0$

$$\Lambda^{top} \mathcal{E}^\vee \simeq K_X \quad \Rightarrow \quad \sum_a n_a^t = \sum_i q_i^t \qquad \forall t$$

$$U(1)^2 \text{ anomaly} \Rightarrow \quad \sum_a n_a^t n_a^s = \sum_i q_i^t q_i^s \quad \forall s, t$$

$$\therefore\ c_1(\mathcal{Z}) - c_1(\mathcal{Z}_1) = \left[\sum_i q_i^t J_t \right] + \left[\sum_i (\vec{q}_i \cdot \vec{d})\, q_i^t J_t \right]$$
$$= c_1(T\mathcal{M}) - c_1(\mathrm{Obs})$$

$$\therefore\ \underline{\Lambda^{top} \mathcal{Z} \otimes \Lambda^{top} \mathcal{Z}_1^\vee \simeq \Lambda^{top} T\mathcal{M} \otimes \Lambda^{top} (\mathrm{Obs})^\vee}$$

exactly as desired.

Not only does GLSM define extension of $\mathcal{Z}, \mathcal{Z}_1$ across compactification divisor, but that extension has good properties.

Another check:

Expect rank $\mathcal{Z}$ − rank $\mathcal{Z}_1$ $= \chi(\phi^*\mathcal{E}) = c_1(\phi^*\mathcal{E}) + \operatorname{rank}\mathcal{E}$

Recall $\quad 0 \to \mathcal{O}^{\oplus m} \to \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\bar{n}_a \cdot \bar{d})) \otimes_{\mathbb{C}} \mathcal{O}(\bar{n}_a) \to \mathcal{Z} \to 0$

$$\mathcal{Z}_1 \simeq \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\bar{n}_a \cdot \bar{d})) \otimes_{\mathbb{C}} \mathcal{O}(\bar{n}_a)$$

Calculate:

$$\operatorname{rank}\mathcal{Z} = \sum_a h^0(\mathbb{P}^1, \mathcal{O}(\bar{n}_a \cdot \bar{d})) - m$$

$$\overset{*}{=} \sum_{\bar{n}_a \cdot \bar{d} \geq 0} (\bar{n}_a \cdot \bar{d} + 1) - m$$

$$\operatorname{rank}\mathcal{Z}_1 = \sum_a h^1(\mathbb{P}^1, \mathcal{O}(\bar{n}_a \cdot \bar{d}))$$

$$= \sum_{\bar{n}_a \cdot \bar{d} < 0} (-\bar{n}_a \cdot \bar{d} - 1)$$

$$\therefore \operatorname{rank}\mathcal{Z} - \operatorname{rank}\mathcal{Z}_1 = \left[\sum_{\bar{n}_a \cdot \bar{d} \geq 0} (\bar{n}_a \cdot \bar{d} + 1) - m\right] - \left[\sum_{\bar{n}_a \cdot \bar{d} < 0} (-\bar{n}_a \cdot \bar{d} - 1)\right]$$

$$= \underbrace{\left[\sum_a 1\right]}_{\operatorname{rank}\mathcal{E}} - m + \underbrace{\sum_a (\bar{n}_a \cdot \bar{d})}_{c_1(\phi^*\mathcal{E})} \quad \checkmark$$

*

Now let's return to the case of reducible bundles, & perform the same checks.

Recall
$$\mathcal{E} = \bigoplus_a \mathcal{O}(\vec{n}_a)$$

$$\mathcal{I} = \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$

$$\mathcal{I}_1 = \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$

Check ranks:

$$\text{rank } \mathcal{I} - \text{rank } \mathcal{I}_1 = \sum_a h^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) - \sum_a h^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d}))$$

$$= \sum_{\vec{n}_a \cdot \vec{d} \geq 0} (\vec{n}_a \cdot \vec{d} + 1) - \sum_{\vec{n}_a \cdot \vec{d} < 0} (-\vec{n}_a \cdot \vec{d} - 1)$$

$$= \underbrace{\sum_a 1}_{\text{rank } \mathcal{E}} + \underbrace{\sum_a \vec{n}_a \cdot \vec{d}}_{c_1(f^* \mathcal{E})}$$

✓

Next, check $c_1(\mathcal{Z} \ominus \mathcal{Z}_1)$:

Recall
$$\mathcal{Z} = \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$
$$\mathcal{Z}_1 = \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a)$$

Thus
$$c_1(\mathcal{Z}) - c_1(\mathcal{Z}_1) = \left[\sum_{\vec{n}_a \cdot \vec{d} \geq 0} (\vec{n}_a \cdot \vec{d} + 1) n_a^t J_t \right] - \left[\sum_{\vec{n}_a \cdot \vec{d} < 0} (-\vec{n}_a \cdot \vec{d} - 1) n_a^t J_t \right]$$
$$= \left[\sum_a n_a^t J_t \right] + \left[\sum_a (\vec{n}_a \cdot \vec{d}) n_a^t J_t \right]$$

S'pose tangent bundle described by
$$0 \to \mathcal{O}^{\oplus k} \to \bigoplus_i \mathcal{O}(\vec{q}_i) \to TX \to 0$$
then the consistency conditions are

$$\sum_a n_a^t = \sum_i q_i^t \quad \forall t \qquad [\Rightarrow \Lambda^{top} \mathcal{E}^\vee \cong K_X]$$
$$\sum_a n_a^t n_a^s = \sum_i q_i^t q_i^s \quad \forall s, t \qquad [\Rightarrow ch_2(\mathcal{E}) = ch_2(TX)]$$
$$\therefore \; c_1(\mathcal{Z}) - c_1(\mathcal{Z}_1) = \left[\sum_i q_i^t J_t \right] + \left[\sum_i (\vec{q}_i \cdot \vec{d}) q_i^t J_t \right]$$
$$= c_1(T\mathcal{M}) - c_1(\mathrm{Obs})$$

Note

Different physical presentations of the same bundle $\mathcal{E}$ can define <u>different</u> extensions of $\mathcal{Z}$ over the compactification divisor.

<u>Ex</u> $X = \mathbb{P}^1$, $\mathcal{E} = T\mathbb{P}^1 = \mathcal{O}(2)$

1) Cokernel presentation
   → explicit (2,2) susy
   → 2 fermi multiplets, charge 1, + 1 neutral chiral

$$0 \to \mathcal{O} \to \bigoplus_1^2 \mathcal{O}(1) \to T\mathbb{P}^1 \to 0$$

induces $$0 \to \mathcal{O} \to \bigoplus_1^2 H^0(\mathbb{P}^1, \mathcal{O}(d)) \otimes_{\mathbb{C}} \mathcal{O}(1) \to \mathcal{Z} \to 0$$

$$\therefore \mathcal{Z} = T\mathcal{M} = T\mathbb{P}^{2(d+1)-1} = T\mathbb{P}^{2d+1}$$

$[c_1 \mathcal{Z} = 0]$

2) Free fermion presentation
   → (0,2) susy
   → 1 fermi multiplet, charge 2

$\mathcal{E} = \mathcal{O}(2)$ induces $\mathcal{Z} = H^0(\mathbb{P}^1, \mathcal{O}(2d)) \otimes_{\mathbb{C}} \mathcal{O}(2)$

$$= \bigoplus_1^{2d+1} \mathcal{O}(2) \neq T\mathcal{M}$$

- same rank, same on noncompact subvariety defined by honest maps
- but <u>different</u> extensions over compactification divisor

More general case: cohomology of a monad

$$\mathcal{O}^{\oplus K} \xrightarrow{A} \bigoplus_a \mathcal{O}(\vec{n}_a) \xrightarrow{B} \bigoplus_i \mathcal{O}(\vec{m}_i)$$

$$\mathcal{E} \equiv \frac{\ker B}{\operatorname{im} A}$$

[Physically, the A map realized in susy trans' of fermi mult's;
" B " " as superpotential $p^i F_i^a \Lambda_a$]

$$(1): \quad 0 \to \ker B \to \bigoplus_a \mathcal{O}(\vec{n}_a) \longrightarrow \bigoplus_i \mathcal{O}(\vec{m}_i) \to 0$$

$$(2): \quad 0 \to \mathcal{O}^{\oplus K} \xrightarrow{A} \ker B \to \mathcal{E} \to 0$$

(2) induces
$$0 \to \mathcal{O}^{\oplus K} \to (\widetilde{\ker B})_0 \to \mathcal{Z} \to 0$$
$$\mathcal{Z}_1 \cong (\widetilde{\ker B})_1$$

(1) induces

$$0 \to (\widetilde{\ker B})_0 \to \bigoplus_a H^0(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a) \to \bigoplus_i H^0(\mathbb{P}^1, \mathcal{O}(\vec{m}_i \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{m}_i)$$

$$\to (\widetilde{\ker B})_1 \to \bigoplus_a H^1(\mathbb{P}^1, \mathcal{O}(\vec{n}_a \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{n}_a) \to \bigoplus_i H^1(\mathbb{P}^1, \mathcal{O}(\vec{m}_i \cdot \vec{d})) \otimes_{\mathbb{C}} \mathcal{O}(\vec{m}_i) \to 0$$

Straightforward to check

$$\operatorname{rank} \mathcal{Z} - \operatorname{rank} \mathcal{Z}_1 = \operatorname{rank} \mathcal{E} + c_1(\phi^* \mathcal{E})$$

$$c_1(\mathcal{Z}) - c_1(\mathcal{Z}_1) = c_1(T\mathcal{M}) - c_1(\mathrm{Obs})$$

The Adams-Basu-Sethi prediction

Consider a deformation of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$, defined by

$$0 \to \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{bmatrix} x_1 & \varepsilon_1 x_1 \\ x_2 & \varepsilon_2 x_2 \\ 0 & \tilde{x}_1 \\ 0 & \tilde{x}_2 \end{bmatrix}} \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \to \mathcal{E} \to 0$$

The group $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee) = \mathbb{C}^2$

Label the generators $X, \tilde{X}$

They predict "heterotic quantum cohomology" relations

$$\tilde{X}^2 = q_2$$

$$X^2 + (\varepsilon_1 - \varepsilon_2) X \tilde{X} = q_1$$

Where did that prediction come from?

One way: 1-loop effective superpotential, following Morrison-Plesser

Can show

$$\tilde{W}_{eff} = \Upsilon_1 \left[i\hat{\tau}_1 - \frac{1}{2\pi} \log\left(\frac{\sigma_1 + \varepsilon_1 \sigma_2}{\Lambda}\right) - \frac{1}{2\pi} \log\left(\frac{\sigma_1 + \varepsilon_2 \sigma_2}{\Lambda}\right)\right]$$

$$+ \Upsilon_2 \left[i\hat{\tau}_2 - \frac{1}{2\pi} \log\left(\frac{\sigma_2}{\Lambda}\right) - \frac{1}{2\pi} \log\left(\frac{\sigma_2}{\Lambda}\right)\right]$$

Then,

$$\frac{\partial \tilde{W}_{eff}}{\partial \Upsilon_a} = 0 \quad \Rightarrow \quad \begin{array}{c} \sigma_2^2 = q_2 \\ (\sigma_1 + \varepsilon_1 \sigma_2)(\sigma_1 + \varepsilon_2 \sigma_2) = q_1 \end{array}$$

which, after a change of variables, is the Adams-Basu-Sethi prediction.

On the (2,2) locus, Morrison-Plesser used same ideas to derive in LSM's Batyrev's conjecture for quantum cohomology of toric varieties.

Outline start of a check of that prediction.

$$\mathcal{M} = \mathbb{P}^{2d_1+1} \times \mathbb{P}^{2d_2+1}$$

For degree $(1,0)$ maps, $\mathcal{M} = \mathbb{P}^3 \times \mathbb{P}^1$.

Let $\alpha_0, \alpha_1, \alpha_0', \alpha_1'$ be homogeneous coord's on $\mathbb{P}^3$

(from expanding x_1, x_2 in zero modes),

let β_0, β_1 be homogeneous coordinates on $\mathbb{P}^1$.

$$0 \to \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{bmatrix} \alpha_0 & \varepsilon_1 \alpha_0 \\ \alpha_1 & \varepsilon_1 \alpha_1 \\ \alpha_0' & \varepsilon_2 \alpha_0' \\ \alpha_1' & \varepsilon_2 \alpha_1' \\ 0 & \beta_0 \\ 0 & \beta_1 \end{bmatrix}} \mathcal{O}(1,0)^4 \oplus \mathcal{O}(0,1)^2 \to \mathcal{Z} \to 0$$

$$\ell \quad \mathcal{Z}_1 \equiv 0$$

Next: correlators

How to usefully represent sheaf cohomology?

- polynomial representatives

$H^*(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee)$:

$$0 \to \mathcal{O} \oplus \mathcal{O} \to \mathcal{O}(1,0)^2 \oplus \mathcal{O}(0,1)^2 \to \mathcal{E} \to 0$$

implies

$$0 \to \mathcal{E}^\vee \to \mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2 \to \mathcal{O} \oplus \mathcal{O} \to 0$$

implies

$$0 \to H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee) \to H^0(\cancel{\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2}) \to H^0(\mathcal{O} \oplus \mathcal{O})$$

$$\to H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee) \to H^1(\cancel{\mathcal{O}(-1,0)^2 \oplus \mathcal{O}(0,-1)^2}) \to \cdots$$

$$\therefore H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee) \simeq H^0(\mathcal{O} \oplus \mathcal{O}) \simeq \mathbb{C}^2$$

So each element of $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee)$ has a <u>constant</u> polynomial representative.
Expand in zero modes to get poly' rep' of $H^1(\mathfrak{M}, \mathcal{Z}^\vee)$, & note for same reasons as above,

$$H^1(\mathfrak{M}, \mathcal{Z}^\vee) \simeq H^0(\mathcal{O} \oplus \mathcal{O}) \simeq \mathbb{C}^2$$

In the degree (1,0) case,

$$\text{rank } Z = \dim \mathcal{M} = 4$$

so we can try to verify this "heterotic quantum cohomology" using 4-pt functions of elements of $H^1(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{E}^\vee)$, which have easily-handled polynomial representatives.

... Details in progress ...

Summary

- described formal heterotic analogue of (2,2) curve counting

- used GLSM's to generate sheaves w/ good properties on compactified moduli spaces

- started to outline verification of prediction of Adams - Basu - Sethi

Open problems

- we've conjectured how the obstruction bundle story generalizes, & even seen that the sheaves $\mathcal{F}_1$, $\mathcal{F}_1$ have the right properties,
  but,
  don't yet know how to explicitly map Atiyah class of $\mathcal{E}$ to an element of

$$H^1(\mathcal{M}_1, \mathcal{F}^\vee \otimes \mathcal{F}_1 \otimes (\mathrm{Obs})^\vee)$$

- mathematical understanding of coupling to worldsheet gravity