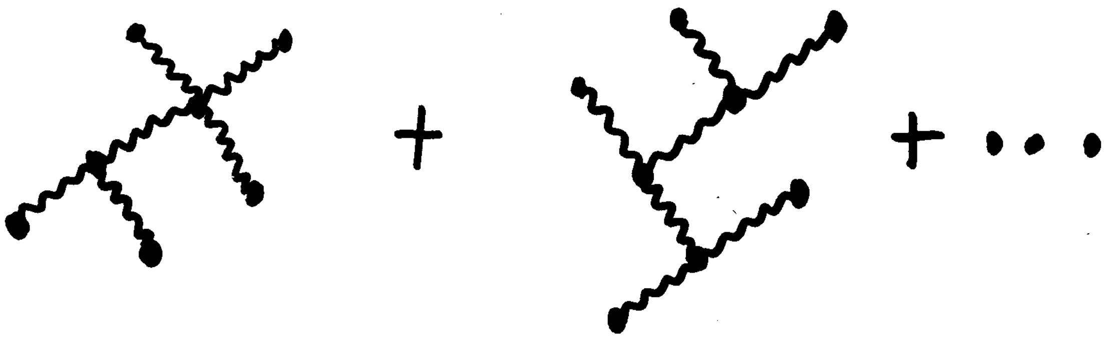
Yang-Mills Amplitudes from String Theory

— in Twistor Space —



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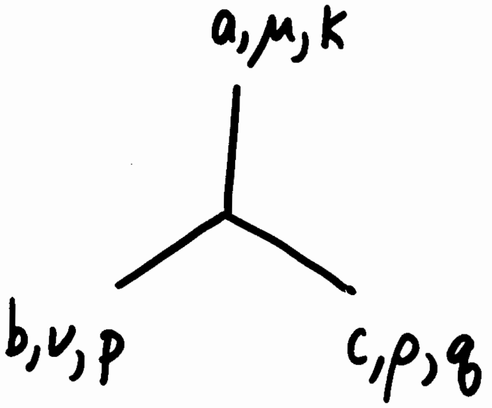
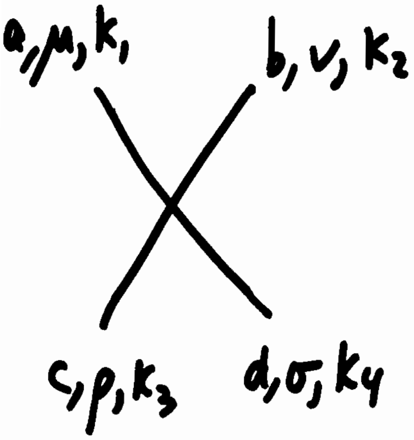
Yang-Mills Theory

Perturbative scattering amplitudes are asymptotic series in the coupling constant g of the theory. We use Feynman diagrams as a convenient (??) mneumonic device for writing down terms in the series.



$$= g f^{abc}\left[g^{\mu\nu}(k-p)^\rho + g^{\nu\rho}(p-q)^\mu + g^{\rho\mu}(q-k)^\nu\right]$$



$$= -ig^2\left[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}\right) + 2 \text{ permutations}\right]$$

$a, b, c, d \in \{1, \dots, \dim(G)\}$

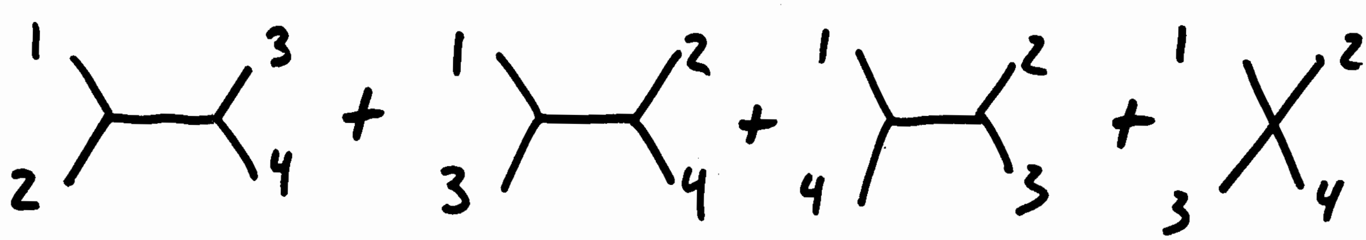
$\mu, \nu, \rho, \sigma \in \{0, 1, 2, 3\}$

Complications

To calculate any amplitude, simply write down all Feynman diagrams and sum them up!

Problems:
- The number of diagrams for an n-particle amplitude grows faster than factorially with n, and
- The Feynman rules are complicated!

For 4 particles, there are 4 diagrams:



36 terms + 36 terms + 36 terms + 8 terms

= a big mess!

There are no obvious cancellations or simplifications of intermediate steps in this calculation.

However, when the dust finally settles...

Spinor Helicity Notation

[Xu, Zhang, Chang]

A gluon scattering amplitude is a function of n momenta p_i^μ and polarizations ϵ_i^μ, but this collection of data is highly redundant:

$$p^2=0 \qquad p\cdot\epsilon=0 \qquad \epsilon^\mu \sim \epsilon^\mu + \alpha p^\mu \ \ \forall\alpha$$

Specifying p^μ and a helicity $+$ or $-$ is almost enough to canonically determine ϵ.

Instead, use the fact that any null vector p^μ can be decomposed as

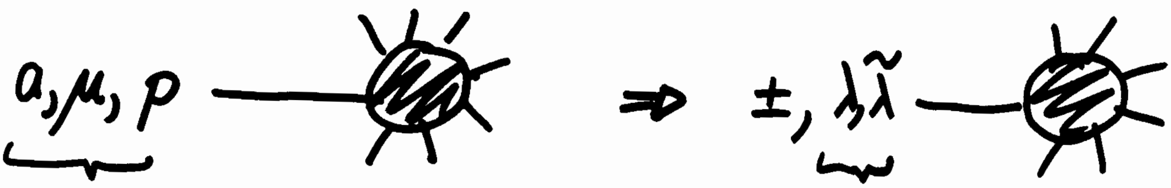
$$p_{a\dot{a}} = p_\mu \sigma^\mu_{a\dot{a}} = \lambda_a \tilde{\lambda}_{\dot{a}}$$

into a pair of (commuting) spinors of opposite chirality.

Moreover, given the pair $(\lambda, \tilde{\lambda})$ and a choice of $+$ or $-$ helicity, there exists a unique (up to gauge transformation) polarization tensor ϵ^μ.

MHV amplitudes

So instead of $A(p_i{}^\mu, \epsilon_i{}^\mu)$ we write $A(\lambda_i^a, \tilde{\lambda}_i^{\dot{a}})$ and specify the helicities of the n gluons.
(We choose wavefunctions which are momentum eigenstates)



specifies a point in an 8+dim(G)-dimensional vector space

specifies a point in a 4-dimensional vector space

This notation allows some very compact formulas.

MHV (maximally helicity violating) amplitudes have 2 negative and $n-2$ positive helicity gluons. The Parke-Taylor formula for MHV is

$$A = \langle \lambda_r, \lambda_s \rangle^4 \cdot \prod_{i=1}^{n} \frac{1}{\langle \lambda_i, \lambda_{i+1} \rangle}$$

(r,s are the two $-$ helicity)

where

$$\langle \lambda_i, \lambda_j \rangle = \epsilon_{ab} \lambda_i^a \lambda_j^b$$

$$[\tilde{\lambda}_i, \tilde{\lambda}_j] = \epsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}}$$

Twistor Space

[For simplicity we work in signature $--++$ where λ and $\tilde{\lambda}$ are independent variables.]

A helicity amplitude can be expressed in twistor space by a Fourier transform

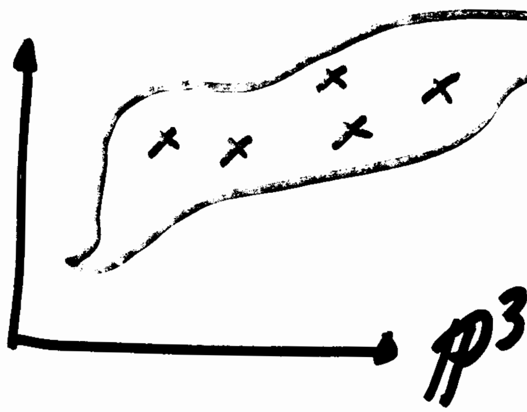
$$A(\lambda_i, \mu_i) = \int \frac{d^{2n}\tilde{\lambda}_i}{(2\pi)^{2n}} \exp[i\,[\tilde{\lambda}, \mu]]\; A(\lambda_i, \tilde{\lambda}_i)$$

The twistor space amplitude is invariant under $(\lambda, \mu) \to (t\lambda, t\mu)$ for any nonzero complex number t. Therefore $A(\lambda_i, \mu_i)$ is not a function of n points on $\mathbb{C}^4$; it is a function of n points on projective space $\mathbb{P}^3$.

This is because the scaling $(\lambda, \tilde{\lambda}) \to (t\lambda, \tilde{\lambda}/t)$ leaves p^μ invariant but scales ϵ^μ as t^2. This cancels the scaling of $d^2\tilde{\lambda}$ in the Fourier transform

Witten's Conjecture

An n-gluon amplitude with q negative helicity gluons, at ℓ loops, is supported on curves in twistor space $\mathbb{P}^3$ of genus $\leq \ell$ and degree $q-1+\ell$.



In other words, A is zero unless there exists a curve of the specified genus and degree which contains all n points.

In particular, MHV amplitudes are supported on lines. — This case is trivial to check directly.

The higher cases of this conjecture imply that amplitudes satisfy some (high order) differential equations. — Checked in several cases.

Differential Equations (Example)

The only degree $d=2$, genus zero curve in $\mathbb{P}^3$ is the plane quadric lying in some $\mathbb{P}^2 \subset \mathbb{P}^3$.

So n points in $\mathbb{P}^3$ lie on some degree 2 curve only if they all lie on some $\mathbb{P}^2 \subset \mathbb{P}^3$.

(For $n \leq 5$ this is if and only if, since 5 or fewer points on some $\mathbb{P}^2$ will generically lie on some conic section.)

Now consider any 4 points Z_i^I $\quad I \in \{0,1,2,3\}$ in $\mathbb{P}^3$. They lie in a common $\mathbb{P}^2$ iff

$$K = \epsilon_{IJKL} Z_1^I Z_2^J Z_3^K Z_4^L = 0$$

It follows that if A_n is the n-gluon amplitude with 3 negative helicity gluons (ie degree 2) then

$$K_{ijkl} A_n = 0 \qquad \forall i,j,k,l \in \{1,\dots,n\}$$

Fourier transforming back $\mu \to \tilde{\lambda}$, K becomes a differential operator.

In all examples that he could get his hands on, Witten found that these kinds of differential eqns. were satisfied.

Destructive $\Rightarrow$ Constructive

The differential equation analysis shows that

$$A_n = \int_{\substack{\text{moduli space of} \\ \text{degree } d \text{ curves in } \mathbb{P}^3}} (\text{something})$$

but we want to <u>calculate</u> (something)

from an honest B-model calculation, and

check that we get the correct answers!

Where does this come from?

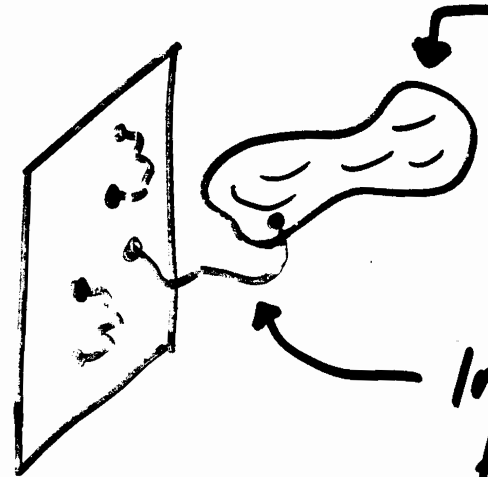
These properties of gluon scattering amplitudes cry out for a deeper explanation. Is there a string theory description in which these properties would be manifest?

Witten proposed the string field theory of the open topological B-model on $\mathbb{P}^{3|4}$ with N D5 branes.

The spectrum of this theory is <u>exactly</u> that of $\mathcal{N}=4$ $SU(N)$ Yang-Mills theory, without the usual infinite tower of extra string states. In some sense α' is already zero in this theory.

...

The main prescription — in pictures



An instantonic I-brane wrapping some holomorphic curve C in $\mathbb{P}^{3|4}$

In an instanton background, there are new degrees of freedom J

N D5 branes

$\Rightarrow$ Integrating out the J degrees of freedom gives a contribution to the effective action for the strings on the D5-branes.

For any fixed C we [redacted] need correlation functions of products of J's on C $\langle J_1 \cdots J_n \rangle$

But C can be anything, so we have to integrate over all possible C.

Curves in $\mathbb{P}^{3|4}$

A curve of degree d is most conveniently parametrized by an embedding of $\mathbb{P}^1$ (a sphere) into $\mathbb{P}^{3|4}$.

$$Z^I = P^I(\omega) = \sum_{k=0}^{d} \omega^k a_k^I$$

$$\psi^A = G^A(\omega) = \sum_{k=0}^{d} \beta_k^A \omega^k$$

parameters of the curve

coordinate on $\mathbb{P}^1$

coordinates on $\mathbb{P}^{3|4}$

To integrate over all possible curves means, practically, to integrate over all of the parameters a_k^I and β_k^A.

Main Formula

The (twistor space Fourier transform of the) tree level n-gluon scattering amplitude with q negative helicity gluons is

$$B(\lambda_i, \mu_i, \psi_i^A) = \int \frac{d^{4q}a \; d^{4q}\beta \; d^n w}{\mathrm{vol}(GL(2))} \langle J(w_1) \cdots J(w_n) \rangle$$

$$\times \prod_{i=1}^{n} \delta^3\left(\frac{Z_i^I}{Z_i^J} - \frac{P^I(w_i)}{P^J(w_j)} \right) \delta^4\left(\frac{\psi_i^A}{Z_i^J} - \frac{G^A(w_i)}{P^J(w_j)} \right)$$

[Roiban, M.S., Volovich]

Note: $\langle J \cdots J \rangle = \prod_{i=1}^{n} \frac{1}{w_i - w_{i+1}}$

Only hard because

- $GL(2)$ acts nonlinearly on a and β
- δ-functions of (polynomial)/(polynomial)

Reduction to Equations

It is convenient to first Fourier transform this. Then half of the moduli integrals are trivial, and we have (omitting femionic piece for now).

$$A(\lambda_i, \tilde{\lambda}_i) = \int \frac{d^{2d+2}a \; d^n\sigma \; d^n\xi}{\mathrm{vol}(GL(2))} \prod_{i=1}^{n} \frac{1}{\xi_i(\sigma_i - \sigma_{i+1})}$$

$$\times \prod_{i=1}^{n} \delta^2(\lambda_i^a - \xi_i \, ?_i^a) \prod_{k=0}^{d} \delta^2\Big(\sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{a}}\Big)$$

$(2d+2) + (n) + (n) - 4 = 2n + 2d - 2$ integration variables

and $2n + 2d + 2$ delta-functions

The equations $\quad 0 = \sum_{i=1}^{n} \xi_i \, \sigma_i^k \, \tilde{\lambda}_i^{\dot{a}} \qquad i = 1, \ldots, n$

$\qquad k = 0, \ldots, d$

$\lambda_i^a = \sum_{k=0}^{d} \xi_i \, \sigma_i^k a_k^a \qquad a, \dot{a} = 1, 2$

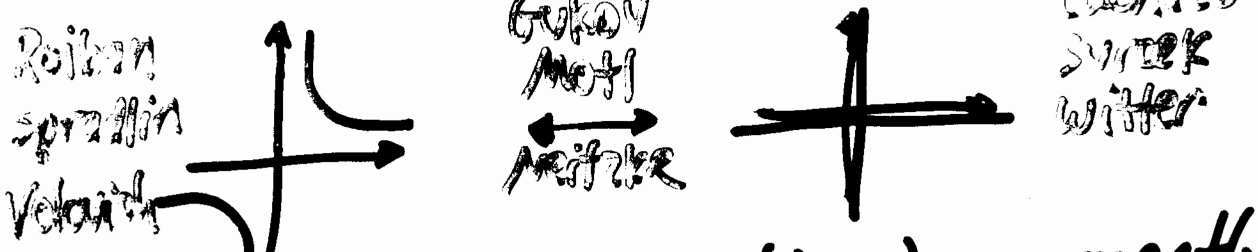
imply momentum conservation $\quad 0 = \sum_{i=1}^{n} \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$

If $\lambda, \tilde{\lambda}$ satisfy this, then there are $2n + 2d - 2$ independent equations.

Integration Contour?

All of the integration variables (the moduli of the curve and the positions on $\mathbb{P}^1$ where the operators are inserted) are complex — to calculate the amplitude we have to specify an integration contour — but the natural contour doesn't work (all variables real).

In Gukov's talk you heard about the relation between the connected curves we studied and the disconnected curves studied by Cachazo, Svrcek & Witten.



These different 'prescriptions' apparently correspond to different choices of integration contour in some huge poorly understood configuration space.

The New Technology

The calculation of an n-particle scattering amplitude reduces to the problem of finding all solutions to the equations

$$\lambda_i^a = \sum_{k=0}^{d} \xi_i \sigma_i^k a_k^a$$

$$0 = \sum_{i=1}^{n} \xi_i \sigma_i^k \tilde{\lambda}_i^{\dot{a}}$$

for ξ_i, σ_i, and a_k^a in terms of $\lambda, \tilde{\lambda}$ and then summing a certain Jacobian over the set of roots using the rule

$$\int d^n x \prod_{i=1}^{n} \delta(F_i(x)) = \sum_{\text{roots}} \left[\det \frac{\partial F_i}{\partial x_j}\right]^{-1}$$

Q_1: What is $N_{n,d}$, the number of roots?

At the moment, all we know is

$N_{n,1} = N_{n,n-3} = 1 \quad N_{6,2} = 4 \quad N_{n,n-d-2} = N_{n,d}$

Parity Invariance

Amplitudes must be symmetric under $\lambda \longleftrightarrow \tilde{\lambda}$ as a consequence of parity [redacted] symmetry. This symmetry is completely obscured in twistor space, since we Fourier transformed $\tilde{\lambda}$ but not λ. Nevertheless we have given an elementary proof of this symmetry (see also a recent paper of Witten). [[illegible] A. Volovich]

Parity interchanges degree d curves and degree $n-d-2$ curves, so its geometrical interpretation is very obscure indeed!

Let me now prove

Theorem. $N_{n,\, n-d-2} = N_{n,d}$

Proof

- Consider $p_m = \sum_{i=1}^{n} \frac{\sigma_i^{\,m}}{\prod_{j\neq i}(\sigma_i - \sigma_j)}$

This is a polynomial in the σ's of degree $m-n+1$.
(It has no poles, and growth at infinity fixes degree.)

- Define $\tilde{\xi}_i = \left[\xi_i \prod_{j\neq i}(\sigma_i - \sigma_j)\right]^{-1}$

- Suppose (σ, ξ) satisfy the equations

$$\lambda_i = \sum_{k=0}^{d} \xi_i \sigma_i^{\,k} a_k$$

Then $$\sum_{i=1}^{n} \tilde{\xi}_i \sigma_i^{\,m} \lambda_i = \sum_{k=0}^{d} a_k\, p_{k+m}$$

$$= 0 \quad \text{for } m \in \{0, \ldots, n-d-2\}$$

(and vice versa)

- Thus $\exists$ 1-1 correspondence between solutions of

$$\lambda_i = \sum_{k=0}^{d} \xi_i \sigma_i^{\,k} a_k \qquad \qquad \tilde{\lambda}_i = \sum_{\ell=0}^{n-d-2} \tilde{\xi}_i \sigma_i^{\,\ell} \tilde{a}_\ell$$

$$0 = \sum^{n} \xi_i \sigma_i^{\,k} \tilde{\lambda}_i \qquad \qquad 0 = \sum_{i=1}^{n} \tilde{\xi}_i \sigma_i^{\,\ell} \lambda_i$$

Conclusion

We've known about QCD for four decades, so you might have thought that

- Every quantity of experimental relevance (that we know how to calculate) had long ago been computed to greater accuracy than the experimental uncertainty, but this is far from the case! (disclaimers...)
- There couldn't possibly remain any undiscovered mathematical structure in tree level amplitudes, but this is far from the case!

- Please help us understand the mathematical structure of the equations.
- What useful mathematical structures appear in loop amplitudes?