Problem #1: The spectral lines of a typical White Dwarf (Sirius B) are observed.

a) Which of the following spectral lines will be prominent (strong):
   i. Fe I
   ii. He I
   iii. Ca II
   iv. Si III
   v. He II
   vi. TiO
   (hint: you do not have to calculate very much here!)

b) One of the observed spectral lines is at \( \lambda = 402.5 \text{ nm} \). Estimate the width of this line due to Doppler Broadening, and also due to natural broadening. (For natural broadening, use \( \Delta t \sim 10^{-8} \text{ sec} \). Assume the emitting atom has atomic mass = 4.

c) If we wanted to measure the magnetic field of the star using Zeeman splitting, how small of a magnetic field could we detect, given the spectral broadening calculated in part b above? How does this compare to the typical magnetic field of a white dwarf like Sirius B? You will be graded both for calculating the size of the Zeeman splitting as well as using a reasonable estimate for the white dwarf magnetic field.
Problem # 2: A white dwarf star (Radius = 0.0073 R\(_\odot\), Mass=1.4M\(_{\odot}\)) goes over the Schoenberg-Chadrasekhar limit and collapses to a neutron star. At that point the collapse stops.

a) Calculate the collapse time. You can assume a free fall for a test mass at the surface of the white dwarf.

b) Calculate the total amount of energy liberated in the collapse.

c) Estimate the luminosity (=energy liberated/collapse time) and bolometric magnitude (from the luminosity), using the results of parts a) and b).

d) This is a Type 1a SN. Type 1a SN have \(<M_\nu> \sim <M_\nu>= -19.3\). Explain why the bolometric magnitude calculated in part c) is so far off from the actual value.

\[ \text{Surface Mass falls a distance } a = R_{\text{wd}} - R_{\text{ns}} \]

\[ \frac{R_{\text{wd}}}{R_{\text{ns}}} = \frac{M_{\text{ns}}}{m_e} \left( \frac{2}{A} \right)^{\frac{2}{3}} = 5.12 \]

\[ t_{\text{ff}} = \left( \frac{3}{32} \cdot \frac{1}{G \rho_0} \right)^{\frac{1}{2}} \]

\[ \text{since } R_{\text{ns}} \ll R_{\text{wd}} \]

\[ t_{\text{ff}} \text{ is essentially collapse to radius } r=0 \]

\[ \frac{\rho_0}{M_\odot} \text{ is the density, so} \]

\[ \frac{1.4M_\odot}{\frac{4}{3} \pi R^3} = \frac{1.4M_\odot}{\frac{4}{3} \pi \left(0.0073\right)^3 \left(6.95 \times 10^8\right)^3} = \]

\[ = \frac{1.4 \left(2 \times 10^{30}\right)}{5.47 \times 10^{20}} = 5.11 \times 10^9 \text{ kg/m}^3 \]

\[ t_{\text{ff}} = 0.93 \text{ seconds} \]

\[ \text{Velocity} = \frac{\text{distance}}{\text{time}} = \frac{(0.0073)(6.95 \times 10^8)}{0.93} \]

\[ \text{Initial } u_{\text{initial}} = -\frac{2}{5} \frac{GM^2}{R_{\text{wd}}} \]

\[ u_{\text{final}} = -\frac{3}{5} \frac{GM^2}{R_{\text{ns}}} = -\frac{3}{5} \frac{(5.12)GM^2}{R_{\text{wd}}} \]

\[ \Delta u = u_{1} - u_{f} = \frac{3}{5} GM^2 \left( \frac{1}{R_{\text{wd}}} \right)(5.12 - 1) = \frac{3}{5} (5.11) \frac{GM^2}{R_{\text{wd}}} = \frac{3(6.4 \times 10^5)(3.8 \times 10^9)5.11}{5(0.0073)(6.95 \times 10^8)} \]

\[ = 3.1 \times 10^{46} \text{ Joules} \]
\[ L = \frac{3.12 \times 10^{46}}{0.93} = 3.3 \times 10^{46} \text{ Watts.} \]

\[ M = M_0 - 2.5 \log_{10} \left( \frac{L}{L_0} \right) \]

\[ = 4.74 - 2.5 \log_{10} \left( \frac{3.3 \times 10^{46}}{3.8 \times 10^{26}} \right) = -45.10 \text{ (yikes!) } \]

This is the total energy liberated. off by 25 optical.

1. Only \( \approx 1\% \) of energy goes into visible. Most goes into neutrinos.
2. Energy lost to kinetic energy (shell) \( \sim 5 \text{ magnitudes.} \)
3. Emission doesn't all leave in 1 second, photon opacity is large enough so emission takes place over hours \( \sim 10 \text{ magnitudes.} \)
Problem #3: In the center of an active galactic nucleus there exists a $10^8 \, M_\odot$ black hole. Material is fed from the surrounding region at a rate of $\frac{dM}{dt} \sim 1 \, M_\odot$/year. This forms an accretion disk around the black hole. It is found that the high energy radiation emitted by this accretion disk is time variable with changes in luminosity on the $\sim 1$ hour time scale.

a) Calculate Schwarzschild radius $R_s$ of this black hole.

\[ R_s = \frac{2m}{c^2} = \frac{2(10^8 \times 2 \times 10^{20})(6.6 \times 10^{-22})}{(3 \times 10^8)^2} = 2.93 \times 10^9 \, \text{m} = 1.96 \, \text{AU} \]

b) So you expect disk to be $\sim 1$ light hour across

\[ = 1.08 \times 10^{12} \, \text{m}. \]  
This neglects General Relativity Effects, which will shrink the radius by

\[ \text{down to about } \sim 2-3 \text{ of that radius.} \]

(c) $T_{\text{max}} = 0.488 \left( \frac{3 \, \text{G} \, m^2}{\pi \, 6 \, R^3} \right)^{1/4}$ (neglecting general) 

\[ = 0.488 \left( \frac{3(6.6 \times 10^{21})(108)(2 \times 10^{30})^2}{5 \pi(3.14)(5.67 \times 10^8)(2.93 \times 10^{11})^3 (60)^3 (24) 365.24} \right)^{1/4} \]

\[ \sim 2.51 \times 10^5 \, \text{K} \]  
As seen next to the disk.  
(no General Relativity)

\[ \lambda_{\text{max}} = \frac{(500 \, \text{nm})(5800 \, \text{k})}{2.5 \times 10^5 \, \text{K}} = 11.6 \, \text{nm} \]

\[ E = \frac{h \nu}{\lambda} = 102.9 \, \text{eV} \]

As seen by someone near the disk. (No GR)
N.B. an external viewer would see a G.R. redshift
for the hot spot at \( r_0 = \frac{4q}{3c^3} R_5 \)

\[ V_{\infty} = V_0 \left(1 - \frac{2Gm}{r_0 c^2}\right)^{1/2}; \quad V_0 \left(1 - \frac{36}{4q}\right)^{1/2} = 0.51 V_0 \]

so wave length doubles, \( \lambda_{\text{max}} \) gets cut in half.

\[ \lambda = \frac{11.6 \text{ nm}}{0.51} = 22.74 \text{ nm} \quad \Rightarrow \quad \lambda_{\text{max}} = 514 \text{ ev} \]

Temperature is characterized by \( \lambda_{\text{max}} \)

\[ T_{\text{max}} = (2.5 \times 10^5) 0.51 = 1.28 \times 10^5 \text{ K} \]

for an outside observer.

If they get \( T_{\text{disk}} \), \( \lambda_{\text{max}} \), near the disk (no Ga) they get full credit.

If they spot the G.R. correction and do it correctly give them an extra 3 points.

If they spot it but do it incorrectly, 1-2 points would be appropriate.
Problem #4: Assume we have a very hot (T=10⁷ K) fireball of completely ionized stellar gas (X = 0.7). The mass of the fireball is 1 M☉ and at time t=0 it is confined to a radius R⊙ = 1 R☉. You can assume it has uniform pressure and temperature throughout, so this is not in an equilibrium situation.

a) Calculate the energy density of the photons in the fireball
b) Calculate the energy density of the hydrogen gas in the fireball.
c) Estimate the ratio of the photon pressure to the hydrogen gas pressure in the star.
d) Starting at t=0, what happens to the fireball? Does it expand or contract and why? (Hint: Think about the Eddington luminosity, or think about our sun!)

(a) \[ U_\gamma = \frac{1}{2} a T^4 = \frac{1}{2} (7.56 \times 10^{-16} \times 10^{20}) = 7.56 \times 10^4 \text{ Joules/m}^3 \]

(b) Single hydrogen energy: \[ \frac{3}{2} k_B T = 1.5(1.38 \times 10^{-23})(10^5) = 2.07 \times 10^{-18} \text{ Joules} = \langle \epsilon \rangle \]

\[ \text{Energy density} = \frac{\text{hydrogen}}{\text{vol}} + \langle \epsilon \rangle = \frac{\sqrt{M_\text{H} \frac{3}{2} \pi R^2 \rho}}{M_\text{H} \frac{4}{3} \pi R^2} \]

\[ = \frac{3(0.7)(2 \times 10^{25})(2.07 \times 10^{-18})}{4(3.14)(6.95 \times 10^8)^3(1.67 \times 10^{-27})} = 1.23 \times 10^{12} \text{ Joules/m}^3 \]

(c) \[ \frac{P_{\text{photon}}}{P_{\text{gas}}} = \frac{\frac{1}{2} U_{\text{photon}}}{\frac{3}{2} U_{\text{gas}}} = \frac{1}{2} \frac{U_{\text{photon}}}{U_{\text{gas}}} = 3.07 \times 10^{-8} \]

So gas pressure very dominant over photon pressure

(d) If I was at the surface of the fireball, it would be the same as being at the surface of the Sun, except a lot hotter. So gas pressure is much higher and so the surface will expand. At the same time the core will begin to contract as mature metals (core temp ≤ core temp in Sun). Overall, it expands and forms a dense core.
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   iv. Si III
   v. He II
   vi. TiO

(hint: you do not have to calculate very much here!)

b) One of the observed spectral lines is at \( \lambda = 402.56 \text{ nm} \). Estimate the width of this line due to Doppler Broadening. Assume the emitting atom has atomic mass \( = 4 \).

c) Calculate the normal Zeeman splitting for this wavelength assuming a 20 T magnetic field. Would this be visible, given the above Doppler broadening?

\[
\text{(c) } \left( \frac{T}{27,000 \text{ K}} \text{ for } \text{Sirius B, so look at Figure 8.11} \right)
\]

He I and Si III are prominent spectral lines at this temperature. Fe I, Ca II, He II, and TiO are not very strong.

\[
D \lambda = \frac{2 \lambda}{c} \sqrt{\frac{2kT_e n_e}{m}}
\]

\[
= \frac{2(402.56 \times 10^{-9})}{3 \times 10^{18}} \sqrt{\frac{1.38 \times 8.72 \times 10^{-16}}{6.69 \times 10^{-27}}} = (2.6 \times 10^{-15}) (8.766) = 2.35 \times 10^{-10} \text{ m}
\]

\[
= 0.0235 \text{ nm}
\]

\[
\text{Splitting in } \nu = \frac{2 \nu_0 \lambda}{\lambda_0} = \frac{eB}{4\pi m}\text{ Reduced mass } = \text{ electron mass}
\]

\[
\nu = \frac{eB}{\lambda_0} \quad d\nu = \frac{eB}{\lambda_0} d\lambda = \frac{eB}{4\pi m} \quad d\lambda = \frac{eB \lambda^2}{4\pi m e c}
\]

\[
\frac{(1.602 \times 10^{-19})(20 \times 402.5 \times 10^{-9})^2}{4(3.14)(9.109 \times 10^{-24})(3 \times 10^{-18})
\]

\[
= 1.5 \times 10^{-10} \text{ m } = 0.15 \text{ nm}
\]

So \( \lambda = 402.56 \pm 0.15 \text{ nm} \). Zeeman split.

Yes, it is much larger than the Doppler Broadening.
Problem # 2: A white dwarf star (Radius = 0.0073 R\(_{\odot}\), Mass=1.4M\(_{\odot}\)) goes over the Chandrasekhar limit and collapses to a neutron star. At that point the collapse stops.

a) Calculate the total amount of energy liberated in the collapse.

b) If the white dwarf was originally spinning with a 1.3 day period, what would the period of the neutron star rotation be?

\[ U_{\text{initial}} = -\frac{3}{5} \frac{G M^2}{R_i} \]
\[ U_{\text{final}} = -\frac{3}{5} \frac{G M^2}{R_f} \]
\[ R_{\text{wd}} = R_i \]
\[ R_{\text{ns}} = R_f \]
\[ \frac{R_{\text{wd}}}{R_{\text{ns}}} = \frac{M N}{m_e} \left( \frac{Z}{A} \right)^{5/3} = 512 \]
\[ \Delta U = \frac{3}{5} G \left(1.4 \right)^2 \left(2 \times 10^{30}\right)^2 \left( \frac{512}{0.0073 \times 6.95 \times 10^8} \right) \]
\[ = \frac{3}{5} \left(6.6 \times 10^{-4}\right) \left(7.84 \times 10^{60}\right) (511) \]
\[ = 3.12 \times 10^{46} \text{ Joules} \]

\[ \omega_i \bar{T}_i = \omega_f \bar{T}_f \]
\[ \frac{2\pi}{T_i} \frac{2}{5} \frac{M}{R_i^2} = \frac{2\pi}{T_f} \frac{2}{5} \frac{M}{R_f^2} \]
\[ T_f = T_i \left( \frac{R_f}{R_i} \right)^2 = \frac{T_i}{(512)^2} = \frac{1.3}{512} = 4.95 \times 10^{-6} \text{ day} \]
\[ T_f = 0.428 \text{ seconds} \]
Physics 3060  
December 15, 2006  

Textbook ok, printout of online relativity notes ok, calculators ok. No other notes allowed.

Problem #3: In the center of an active galactic nucleus there exists a $10^8 \, M_\odot$ black hole. Material is fed from the surrounding region at a rate of $dM/dt \sim 1 \, M_\odot$/year. This forms an accretion disk around the black hole. Assume the outer radius of the accretion disk is at a radius of $R = 10$ AU.

a) Calculate Schwarzschild radius $R_S$ of this black hole.

b) Calculate the accretion disk peak temperature $T_{\text{max}}$.

c) Assume the disk radiates as a black body with temperature $T_{\text{max}}$. Calculate the peak wavelength of photons emitted at the peak temperature of the disk.

\[ R_S = \frac{2GM}{C^2} = \frac{2 \times (10^8)(2 \times 10^9)(6.6 \times 10^{-11})}{(3 \times 10^8)^2} = 2.93 \times 10^{11} \text{m} = 1.96 \text{AU} \]

\[ T_{\text{max}} = 0.468 \left( \frac{3 \times 10^3 M}{8\pi G R^3} \right)^{\frac{1}{4}} = 0.468 \left( \frac{3 \times (6.6 \times 10^{-11})(10^8)(2 \times 10^3)}{8 \times (3.14)(5.17 \times 10^8)(2.93 \times 10^{11})} \right)^{\frac{1}{4}} \approx 2.51 \times 10^5 \text{K} \]

Neglecting General Relativity effects, this is the temperature seen by someone next to the hot spot.

\[ \lambda_{\text{max}} = \frac{500 \text{ nm}}{2.5 \times 10^{15}} = 11.6 \text{ nm} \]

\[ \frac{hc}{\lambda_{\text{max}}} = 102.9 \text{ eV} \] for UV very hard UV

as seen by somebody near the disk (no $C - 12$)

\[ \text{N.B. an external viewer would see a $C - R$ shift for the hot spot at } R = \frac{14 \, R_S}{36} \]

\[ \nu_0 = \nu_0 \left(1 - \frac{2GM}{Rc^2}\right)^{\frac{1}{2}} = \nu_0 \left(1 - \frac{36}{49}\right) = 0.51\nu_0 \]

So wavelength doubles $\lambda_{\text{max}} = 22.74 \, \text{nm}$, $\nu = 51.4 \, \text{eV}$

$T_{\text{max}}$ is characterized by $\lambda_{\text{max}}$. $T_{\text{max}} = 0.51(2.5 \times 10^5) = 1.28 \times 10^5 \text{ K.}$ for outside observer.
Grady: If they get Tdisk, lmax using no g.r. (near the disk) they get full credit. If they spot the G.R. connection and do it correctly give them extra 3 points. If they spot it and do it incorrectly give them 1-2 points.
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a) Calculate the energy density of the photons in the fireball.
b) Calculate the energy density of the hydrogen gas in the fireball.
c) Estimate the ratio of the photon pressure to the hydrogen gas pressure in the star.

\[ \epsilon = \frac{1}{6} \alpha T^4 = \frac{1}{6} (7.56 \times 10^{-16}) (10^3) = 2.56 \times 10^{-12} \text{ Joules/m}^3 \]

\[ \text{Single Hydrogen energy} = \frac{3}{2} kT = (1.5)(1.38 \times 10^{-23})(10^3) = 2.07 \times 10^{-18} \text{ Joules} = \epsilon \]

\[ \frac{\epsilon_{\text{hydrogen}}}{\sqrt{1 - \frac{\epsilon}{\epsilon_{\text{hydrogen}}}}} \]

\[ Y = \frac{X(M_{\odot})}{\frac{4}{3} \pi R_0^3 M_{\odot}} (\epsilon) = 3 \left( \frac{0.7(2 \times 10^9)}{2.07 \times 10^{-18}} \right) \frac{4(3.14)(6.95 \times 10^8)^3 (1.67 \times 10^{-27})}{123 \times 10^{12}} \]

\[ U_{\text{gas}} = 1.23 \times 10^{12} \text{ Joules/m}^3 \]

\[ \frac{P_{\text{photon}}}{P_{\text{gas}}} = \frac{1}{3} \frac{U_{\text{photon}}}{U_{\text{gas}}} = \frac{1}{2} \frac{U_{\text{photon}}}{U_{\text{gas}}} = \frac{2.07 \times 10^{-8}}{3.07 \times 10^{-8}} \]

So gas pressure is the dominant force, by far.