A geometric interpretation of symmetry fractionalization

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Our goal in these notes is to uncover the geometric meaning of symmetry fractionalization. This is useful because it fits in nicely with our field theory plan of attack, and because of the general “the more ways of understanding something, the better” principle. Moreover, I haven’t seen anything like this in the literature, which is a bit surprising since I think the correspondence is very natural and gives us a nice physical picture for how fractionalization works. It goes without saying, but I am going to prioritize a physical understanding of this issue above a mathematical one. As a result there will be a lot of heuristic explanations and imprecise statements, but in reality a rigorous understanding of this isn’t very far below the surface.

From our algebraic treatment of symmetry fractionalization, we know that the different ways in which the symmetry group $G$ can fractionalize over the group $N$ are given by the different kinds of ways to extend $N$ over $G$, each of which are represented by a short exact sequence

$$1 \to N \to E \to G \to 1.$$  \hspace{1cm} (1)

This exact sequence viewpoint is seemingly a purely algebraic concept. However, in our travels we’ve encountered a few examples which suggest that symmetry fractionalization is somehow related to a “twisting” of different spaces, which has a distinctly geometric character. For example, when the gauge fields in 1+1 D theories carry fractional quantum numbers, they are anomalous, and must be coupled to a 2+1 D bulk theory. That bulk theory must be a Chern-Simons type theory, and the Chern-Simons term, being related to the Chern class, must involve some sort of “twisting” of a bundle over the gauge field. Similarly, we will show in a later set of notes that when a gauge field $a \in C^1(X, N)$ transforms projectively under $G$, it usually transforms as $a \mapsto a + \gamma \cup \delta \gamma$, for $\gamma \in C^0(X, N)$ (here $X$ is the spacetime manifold). We again get a Chern-class-like term appearing, which suggests that there’s a twisted bundle hiding somewhere in the theory. How do we unify our algebraic understanding of fractionalization with these geometric thoughts?

The way to think about things geometrically is to realize that an extension of $G$ by $N$ is actually the same thing as a fiber bundle $E$ with base space $G$ and fibers given by $N$. To put it another way, when we extend $G$ by $N$ we give each $g \in G$ its own copy of $N$, which is the same thing as creating a fiber bundle $E$ with $G = E/N$. Using this fiber bundle picture, we can make our guesses about fractionalization more precise and understand exactly how fractionalization can be understood as the “twisting” of a bundle.

First, let us suppose that the extension is trivial, so that all representations of $G$ are linear. Recall that the extension of $G$ by $N$ is trivial iff the bundle is a semi-direct product of $N$ and $G$: $E = N \rtimes G$ (regardless of the choice of action of $G$ on $N$). Having $E$ be a semi-direct product is equivalent to saying that the sequence

$$1 \to N \to E \xrightarrow{i} G \to 1.$$  \hspace{1cm} (2)

is split. Saying that the sequence splits means that there exists a map $s : G \to E$ such that the composition $\pi \circ s$ is the identity on $G$. That is, the sequence splits if the projection $\pi$ of the fiber bundle $E$ onto the base space $G$ is invertible, with inverse given by $s$. The map $s$ is a global section: it lifts $G$ up into a cross-section of the fiber bundle, and its invertibility means that projecting this cross-section back down onto $G$ with $\pi$ can be done without the loss of any information. This is saying that if the extension is trivial, the fiber bundle $E$ admits a global section. Now, global sections can only be well-defined if the fiber bundle $E$ is free of “twistiness”, since a global section is essentially a flat slice through the fibers of $E$. Pictorially, the setup looks like this:
If the extension is trivial (and hence splits), $s$ will lift the solid wiggly line indicating the base space $G$ up to another wiggly line $s(G)$ that is essentially “parallel” to the first, as indicated by the dashed line in the above figure. That is, we can lift the base space $G$ to a copy of itself that looks the same as the original. If the extension is not trivial, this is impossible. In this case $E$ is not a semi-direct product, the sequence does not split, no such map $s$ exists, and the base space $G$ must become “twisted” when it gets lifted up into the fibers. That is, $s(G)$ is not “parallel” to $G$, and the dashed line in the above figure becomes twisted. Since the trivial extension is associated with a bundle that admits a global section and is therefore “untwisted”, we see that linear representations correspond to “untwisted” bundles. Alternatively, we can say that projective representations measure the obstruction to lifting a given representation to a linear (untwisted) one.

In fact, we already knew that projective representations are twisted versions of regular representations, since we can define multiplication on $E$ explicitly though

$$(a,g) \cdot (b,h) = (a + g \cdot b + \omega(g,h), gh), \quad a,b \in N, \ g,h \in G,$$

which is a twisted version of a regular group action. As we see from the above equation, the factor set $\omega$ acts as a “perturbation” to the semi-direct product structure, which is the cause of the “twistiness” in the fiber bundle picture. The fiber bundle picture thus shows how this algebraic twisting manifests itself in a geometric way.

This leads us to wonder about the geometric meaning of the factor set. I claim that everything we need to know about the geometry of the projective representations and the geometrical meaning of the factor set can be summed up by the following diagram:

Okay, what’s going on here? We’ve drawn in three of the fibers of the base space, associated with the group elements $g, h, gh \in G$. We let $\Gamma_a(g)$ denote the projective representation of $g$ on the “quasiparticle” $a \in N$, which is located at some point in the fiber over $g$. We do the same for $\Gamma_a(h)$, and then ask what happens when we multiply $g$ by $h$ in the base space. Multiplication in the base space induces a transformation on the fibers, and we end up with $\Gamma_a(gh)$ on the fiber over $gh$. Now in general, this may not be equal to what it is “supposed” to be, namely $\Gamma_a(g)\Gamma_a(h)$. However, since $N$ is a group, we must be able to move along the fiber from $\Gamma_a(gh)$ to $\Gamma_a(g)\Gamma_a(h)$ by multiplying $\Gamma_a(gh)$ by some element in $N$. As indicated in the figure, this element is, by definition, the factor set $\omega_a(g,h)$. If the factor set is trivial, multiplication in the fiber behaves just as you would expect – there is no twisting, and the $\Gamma$s are linear representations. However, nontrivial factor set means that movement around in the base space maps to twisted movement in the fibers – precisely why we said that nontrivial extensions “twist” the sections of the fiber bundle in a nontrivial way.

Thus, we see that if the representation is linear, and hence a global section $s$ exists, the representations $\Gamma(g)$ lift the base space up to a “straight line” in the fiber bundle – i.e., given two points on the line with “$x$” coordinates $g,h$, the point on $s(G)$ with $x$-coordinate $gh$ can be obtained simply by multiplying $\Gamma(g)$ with $\Gamma(h)$. If the representation is projective, the image of $G$ under $s$ is not a “line” in $E$, but becomes some sort of nonlinear thing. This means that moving along $s(G)$ can’t be done simply by regular multiplication – instead, we need to use a multiplication law that is “perturbed” by $\omega$, preventing any prospective section $s$ from being a homomorphism.

One final remark. When we think of anomalies in terms of projective representations, we see that an anomaly occurs when there is an obstruction to taking a local section and making it global. In contrast, when we take the usual approach and think about anomalies as an obstruction to gauging a symmetry, we are dealing with an obstruction to making a global symmetry local. Since these two approaches yield the same results, we have a sort of “local-global” duality. Of course, the story isn’t as simple as I am making it out to be, because we can have some projectivity in the representations without causing an anomaly (some nontrivial fractionalization classes are non-anomalous). Interestingly, we have seen in other notes that anomalies occur when both electric and magnetic sectors have fractional quantum numbers – i.e., anomalies are only possible when we retain some form of electromagnetic duality, the idea.
being that in these cases we can use this duality to convert from the projective representation approach to the gauging approach.

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