A geometric take on symmetry fractionalization

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The idea of these notes is to try to find a nice geometric interpretation of symmetry fractionalization in terms of curvature in configuration space by using simplicial cohomology rather than group cohomology. This idea is sketched out, and allows us to get a nicer feel for the geometric meaning behind anomalies.

I. FRACTIONALIZATION AS CURVATURE IN THE CAYLEY GRAPH, AND EXAMPLES

A. Motivation and general comments

When we think about the role that cohomology plays in classifying symmetry fractionalization, or even the role in plays in the study of topological phases in general, we usually look at it from an algebraic viewpoint – this is reasonable of course, since we’re usually dealing with group cohomology and finite groups. However, I think of cohomology as an inherently geometrical concept (even to compute group cohomology, we always have to go to classifying spaces at some point!), and so I thought it might be nice to see if symmetry fractionalization could be recast in a more geometric fashion.

To see how I got interested in this, we will look at a simple example, where the symmetry group in question is $\mathbb{Z}_2$. We can draw a sort of “Cayley graph” – call it $M_{\mathbb{Z}_2}$ – for the group action as follows:

$$M_{\mathbb{Z}_2} = g \rightarrow g.$$  \hfill (1)

We can imagine that the $\mathbb{Z}_2$ symmetry acts on a quasiparticle $a$ by moving it around the graph by following the arrows. If the fractionalization of $a$ is nontrivial, moving $a$ once around the graph will not produce the same state that we started with. If we compare the processes of acting on $a$ twice with the $\mathbb{Z}_2$ symmetry to parallel transporting $a$ around a closed loop in configuration space (which is what it looks like is happening on the paper), then we can interpret the fractionalization of $a$ as the curvature that $a$ sees in the Cayley graph. If we were to go out on a limb and treat the Cayley graph as some nice continuous space and wanted to examine its curvature, what should we do? Well, from our knowledge of differential geometry, we know that we should compute the de Rham cohomology of the Cayley graph – for us, this is the second de Rham cohomology group. Of course, we are familiar that fractionalization classes are computed by $H^2(G, A)$ (with $A$ the fusion algebra of the theory), and so this suggests that this geometric take on things might not be too misguided.

Saying that quasiparticles experience curvature as they move around $M_G$ is equivalent to saying that certain loops in $M_G$ are threaded with fluxes that the braid nontrivially with the quasiparticles in the theory. That is, we are led to associate each closed loop in $M_G$ with a quasiparticle in the fusion algebra – which is precisely what we did to classify fractionalization from the regular algebraically-focused viewpoint.

We can state this idea more precisely as follows. In order to understand the symmetry fractionalization of a symmetry group $G$ over the quasiparticles described by the fusion algebra $A$, our first task is to construct a $M_G$, a manifold version of the symmetry group $G$ (a “Cayley graph” or a continuous version of a group digraph). We then it into a “fiber bundle” $E$ by giving each point of $M_G$ a fiber of the fusion algebra $A$. If the fibers $A$ on a given $A$-bundle $E$ are finite groups (which is the case here), then the connection on $E$ must be flat. Formally, this is simply because $E$ has to look locally like a product space, and so the values on the fibers as we move across the base space must change continuously, which is impossible if the fibers are discrete. But physically, it of course must be flat – we should expect the theories we’re dealing with to be described by the Chern-Simons action, which being topological requires the connection to be flat (check the eom!). This means that any curvature in $E$ comes entirely from the topological part of the curvature (the monodromy), while the geometric part vanishes. To say it another way, homotopic paths in $M_G$ obviously represent the same physical action on $|\Psi\rangle$, and as such should give the same fractionalization classes. That is, the curvature for path in $M_G$ must be homotopy invariant, and so the geometric part of the curvature must vanish.
In mathematical terms, we would say that the connection is a functor $K : PM_G \to A$ from the group(oid) of paths in $M_G$ to $A$. We aren’t interested in all flat connections though, since as always we must mod out by gauge transformations. For us, a gauge transformation $\zeta$ between two flat connections is, in fancy terms, a natural isomorphism (a morphism of functors) between two connections $K$ and $K'$, and looks like:

\[ \begin{array}{ccc} PM_G & \xrightarrow{K} & A \\ \downarrow \quad & & \downarrow \quad \\ K' & \xrightarrow{K'} & A \end{array} \]  

(2)

Since $\zeta$ needs to map flat connections to flat connections, it must have the property

\[ \zeta(g)\zeta(h) = \zeta(gh), \]  

(3)

which we saw was the defining property of a gauge transformation in the last set of notes (for simplicity, we are assuming that the action of the symmetry group doesn’t permute the quasiparticle species). This is just saying that $\zeta$ is trivially flat everywhere – “parallel transporting” it around any loop in the Cayley graph is trivial, as it can’t detect any of the topology of the space it’s moving around in. In these terms, we thus see that the set of fractionalization classes $\mathcal{F}$ is given by

\[ \mathcal{F} \cong \text{Hom}(\pi_1(M_G), A), \]  

(4)

modulo the trivially flat gauge transformations discussed above.

B. Examples

Let’s see how this works at a more concrete level, starting with the group $\mathbb{Z}_2$. As before, we draw the same single-loop Cayley-graph-manifold, and want to compute the curvature that quasiparticles see in the graph, or equivalently, the quasiparticle flux living in the closed loop of the graph. By looking at $M_{\mathbb{Z}_2}$ and boldly turning it into a continuous manifold, we are prompted to make the identification

\[ M_{\mathbb{Z}_2} = \begin{array}{c} g \\ g = \mathbb{R}P^2 \end{array} \]  

(5)

To get the different possible fractionalization classes (the curvature of the space), we will (seriously) compute the simplicial cohomology of $\mathbb{R}P^2$. If this is to make sense, then we’d better have $H^2(\mathbb{R}P^2; A) = A/2A$, since $H^2(\mathbb{Z}_2, G) = A/2A$. For me, the easiest way to compute the cohomology of $\mathbb{R}P^2$ is by first finding the homology, which is easily done by making use of the cell complex structure in $\mathbb{R}P^2$ (one cell in each dimension, attached through the 2:1 antipodal map, or just use the cell structure drawn in the picture of $M_{\mathbb{Z}_2}$): we find $H_0(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}$, $H_1(\mathbb{R}P^2; \mathbb{Z}) = \mathbb{Z}_2$, and $H_2(\mathbb{R}P^2; \mathbb{Z}) = 0$ (of course, all higher homology groups are zero). To get the cohomology groups, we use the universal coefficient theorem, which gives an exact sequence

\[ 0 \to \text{Ext}(\mathbb{Z}, H_{n-1}(X; A)) \to H^n(X; A) \to \text{Hom}(H_n(X; A), A) \to 0, \]  

(6)

which holds for any topological space $X$ and finite group $A$. For computations involving cell complexes with only 0, 1, and 2-cells, we will usually have $\text{Ker}d_2 = 0$, and so since $\text{Im}d_3 = 0$ as there are no 3-cells, we have $H_2(X; A) = 0$ which implies $H^2(X; A) \cong \text{Ext}(H_1(X; \mathbb{Z}), A)$. In particular, we can use

\[ \text{Ext}(\mathbb{Z}_2, A) = A/nA \]  

(7)

to derive $H^2(\mathbb{R}P^2, A) = A/2A$, which tells us that $\mathbb{Z}_2$ symmetry fractionalization is given by factor sets in $A/2A$, which is consistent with our previous understanding. It’s pretty remarkable that we can recover this result by using the simplicial cohomology of a continuous topological space like $\mathbb{R}P^2$.

The fact that $\mathbb{R}P^2$ turns up here isn’t so surprising when we think about the relationship between group cohomology and regular cohomology of a topological space. As a reminder, the group cohomology $H^n(BA, G)$ for finite groups is secretly the simplicial cohomology of a different group $BG$:

\[ H^n(BA, G) \cong H^n(A, G), \]  

(8)
where $BA$ is the classifying space of $A$, which is a genuine topological space on which we can do simplicial cohomology. $BA$ is defined as the base space of a principal $A$ bundle $EA$ (the universal bundle) with the property that any principal $A$ bundle $E$ over any manifold $M$ allows a bundle map into the universal bundle, and that any two such maps are homotopic. If we don’t want to think about fiber bundles, we can just define $BA$ as the topological space such that $H^n(BA, G) \cong H^n(A, G)$. That such a space always exists is kind of amazing, and in general $BA$ is quite complicated. For example, $B\mathbb{Z}_2 = \mathbb{R}P^\infty$, the infinite-dimensional real projective space. Even though this seems complicated, it really isn’t, since the cell structure is still just one cell in every dimension. This tells us that the appearance of the real projective plane when classifying $\mathbb{Z}_2$ symmetry fractionalization isn’t so surprising, since we just take the first two dimensions of $\mathbb{R}P^\infty$.

We can apply this cell complex oriented thinking to derive the second cohomology groups for the 2D crystal symmetry groups, which are otherwise kind of tricky to calculate. The pure $\mathbb{Z}$ symmetry groups, which are otherwise kind of tricky to calculate. The pure $\mathbb{Z}$ symmetry fractionalization isn’t so surprising, since we just take the first two dimensions of $\mathbb{R}P^\infty$. Now if $g$ and $h$ are related to one another by reflection, and thus we only need to classify one of them, whereas if $h$ is even we need to classify both. Furthermore, if the two mirror planes are related by reflection, then we don’t have any control over the fractionalization of the commutation relation between the two, and so the entire fractionalization pattern is determined by the fractionalization of one of the mirror planes. Each independent mirror plane contributes a factor of $H^2(\mathbb{R}P^2, A) = A/2A$, and for even $h$ the fractionalization of the relation $(ab)^h = 1$ gives a factor of $H^2(\mathbb{Z}_h; A) = A/hA$. Thus, we have

$$H^2(D_{2h}; A) = (A/2A)^{\oplus(2-|h|_2)} \oplus (A/hA)^{\oplus(1-|h|_2)},$$

where $[h]_2 = h \mod 2$. What’s interesting about this is that the fractionalization classes of even $h$ dihedral groups for $A = \mathbb{Z}_2$ (the toric code case) are all the same. Furthermore, all the fractionalization classes consist of direct sums of $\mathbb{Z}_2$’s, suggesting that only relations of the form $g^2 = 1$ can be fractionalized. This is actually expected if we look at the Cayley graph from a topological point of view. Suppose we try to find the fractionalization of the representations $\Gamma_g$ and $\Gamma_h$, with $gh \neq 1$. We want to find the factor set $\alpha(g, h)$, and so we draw the diagram

Now if $g \neq h$ we can’t do any identification, and so the diagram is topologically just a disk, and since $H^2(D^2, A) = 0$ we can’t have a nontrivial $\alpha(g, h)$. If $g = h$ we can identify two of the edges, and topologically the diagram becomes
some sort of weird half projective plane (let’s call it $X$). The cohomology is easy to calculate, though – we have one 0-cell, two 1-cells, and one 2-cell, and so we get $H_0 = H_1 = \mathbb{Z}$, $H_2 = 0$, implying that $H^2(X; \mathbb{A}) \cong \text{Ext}(\mathbb{Z}, \mathbb{A}) = 0$, and so $\alpha(g, g)$ must be trivial as well if $g^2 \neq 1$. Of course when $g^2 = 1$ we have $X = \mathbb{R}P^2$ and get $\alpha(g, g) \in A/2A$ like before.

It would great if I had another way of checking this (other than GAP, which agrees with me), but I haven’t been able to figure out a (simple) analogue for the Kunneth formula for the cohomology of $X \times Y$ rather than $X \times Y$ that doesn’t involve awful spectral sequence calculations.

Now let’s add in translation symmetry. If we have just translational symmetry then $G = \mathbb{Z} \times \mathbb{Z}$ and a few short exact sequences tell us that

$$H^2(\mathbb{Z} \times \mathbb{Z}; A) \cong \text{Hom}(\mathbb{Z}, A) \cong A.$$  

When we have a dihedral group as part of the symmetry as well, adding in translation has the effect of creating a gigantic lattice of Cayley graphs... (to be continued!)

\section*{II. COMMENTS ON ANOMALOUS FRACTIONALIZATION}

This sort of geometric mentality allows us to get a different perspective on anomalous fractionalization, and why anomalies force us to go to higher dimensions. For any three elements $g, h, k \in G$, we can form the tetrahedron

![Tetrahedron Diagram](image)

Now, the 2-cocycle relation reads

$$\alpha_{\rho_a(a)}(h, k)\alpha_a(g, hk) = \alpha_a(g, h)\alpha_a(hk, k).$$  

(14)

Forgetting about $\rho$ for now, we see that this relation tells us that the flux flowing into the tetrahedron through the faces bordering the $gh$ 1-cell is equal to the flux flowing out the tetrahedron through the faces bordering the $hk$ 1-cell – that is, the 2-cocycle relation says that the total flux flowing into the tetrahedron must be zero. The tetrahedron doesn’t have to be flat locally, but it should be globally flat, in that the connection can’t detect the three-dimensional hole in the interior of the tetrahedron. If the 2-cocycle relation isn’t satisfied, there must be a monopole living inside the tetrahedron, and so the 3-cells must play a role in the symmetry fractionalization process, forcing us to treat the system as three-dimensional. Alternatively, we can interpret the 2-cocycle relation as the ability to squash the tetrahedron flat by erasing one of the diagonal lines in equation (14) – the 2-cocycle relation says that both ways the system as three-dimensional. Alternatively, we can interpret the 2-cocycle relation as the ability to squash the tetrahedron into two dimensions are equivalent. This means that if the 2-cocycle relation isn’t satisfied, there is an obstruction to treating the system as purely two-dimensional, and so the resulting symmetry fractionalization patterns can only be realized in 3D, confirming the result we obtained rather formally in the last set of notes. Since $d^2 = 0$, these anomalous fractionalization patterns are classified by $H^3(G, \mathbb{A})$ (of course this is true, since $H^{d+1}$ always represents obstructions to $H^d$).

When the action of $\rho$ is nontrivial, then we can test for the anomaly by gluing the tetrahedra given by $d_\rho\alpha_a(g, h, k)$ and $d_\rho\alpha_{\rho_a(a)}(g, h, k)$ together along the $(h, k, hk)$ face (assuming $\rho_a$ is an involution – if not, more than two tetrahedra have to be glued together and the picture is harder to visualize). To make the gluing work, we first have to “invert” one of the tetrahedra by reversing the direction of all the fluxes passing through its 2-cells. This means that non-anomalous fractionalization classes can still have monopoles living in both the $a$ and $\rho_a(a)$ tetrahedra, as long as these monopoles fuse to the vacuum.

We can use this argument to see why symmetry fractionalization in $\mathbb{Z}_2$ gauge theory is fixed to be trivial when the action $\rho : \mathbb{Z}_2 \to (TC)$ permutes $e$ and $m$ and the $H^3$ obstruction vanishes (i.e. when $d_\rho = 1$), which we derived in the last set of notes. Consider the two tetrahedra given by $d_\rho\alpha_e(g, h, k)$ and $d_\rho\alpha_m(g, h, k)$. The symmetry fractionalization of $e$ or $m$ will be nontrivial when nontrivial fluxes pass through the faces of the respective tetrahedra, making the tetrahedra possess local curvature. Since $d_\rho\alpha_e = d_\rho\alpha_m = 1$ as the obstruction vanishes by assumption, the total flux flowing out of each tetrahedron must vanish, and the connection must be globally flat.
III. USING COHOMOLOGY TO ADD IN SPIN SYMMETRY

Since most of the systems we’re interested in will presumably come with some amount of spin symmetry, it would be nice to understand how the presence of spin symmetry (and maybe $U(1)$ symmetry as well) affects the fractionalization classification. This is actually very easy to do, since we can usually set the total symmetry of the system to just be a direct product between the spin symmetry and whatever other symmetry the system comes with, and so we just need to compute $H^2(G \times G_{\text{spin}}; A)$. This is easily accomplished using the group cohomology version of the Kunneth formula, which gives an exact sequence

$$0 \to \bigoplus_{j+k=n} H^j(X; A) \otimes_R H^k(Y; A') \to H^n(X \times Y, A \oplus_R A') \to \bigoplus_{j+k=n+1} \text{Tor}^R(H^j(X; A), H^k(Y; A')) \to 0,$$

where $R$ is the ring we’re working over, with $A$ and $A'$ thought of as $R$-modules. The above formula only works when at least one of $A$, $A'$ is $R$-free. However, it also has the side benefit that the sequence always splits. Usually we will set $R = \mathbb{Z}$, but depending on what $A$ and $A'$ are sometimes the choice $R = \mathbb{Z}_p$ for prime $p$ is more efficient since fields have no torsion.

To get started, we need to know what $H^*(G_{\text{spin}}; \mathbb{Z})$ is, since we will be using the Kunneth sequence with $X = G$, $Y = G_{\text{spin}}$, $A$ (usually) some $\mathbb{Z}$-module, and $A' = \mathbb{Z}$ so that $A'$ is (trivially) $\mathbb{Z}$-free. This is actually pretty tricky. For example, we might think that getting $H^*(SO(3); \mathbb{Z})$ is easy since $SO(3)$ is diffeomorphic to $\mathbb{R}P^3$, but we need the group cohomology of $SO(3)$, not the regular simplicial cohomology. That is, we need to calculate the cohomology of $BSO(3)$, which is a huge pain and needs to be done with nasty spectral sequences. The same is true for $O(2) = U(1) \rtimes \mathbb{Z}_2$.

Luckily, other people have done it, and they find

$$H^n(O(2); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}_2, & n=1,2,3 \end{cases}, \quad H^n(SO(3); \mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0 \\ \mathbb{Z}_2, & n=1,2,3 \end{cases} \quad (17)$$

So then Kunneth tells us that

$$H^2(G \times SO(3); A) \cong H^2(G; A) \oplus 2A^G,$$

where $2A$ is the 2-torsion subgroup of $A$ and $A^G$ is everything in $A$ fixed under the action of $G$. This makes sense, since $H^2(G \times SO(3); A)$ has a contribution from $H^2(G; A)$ and a choice of projective representation for the spins, but has no terms deriving from the breaking of commutativity of spin rotations with $G$ operations, which is to be expected from the connectedness of $SO(3)$. For $O(2)$ we get the slightly messier

$$H^2(G \times O(2); A) \cong H^2(G; A) \oplus 2A^G \oplus (A^G/2A^G) \oplus 2H^1(G, A),$$

which is the same as the result for $SO(3)$ but with two extra terms. One of them (presumably the $A^G/2A^G$ term) likely derives from the group Hom($O(2), A$) no longer being trivial since $O(2)$ has two connected components. I’m less sure about the interpretation of the $H^1$ term.

We might as well mention what happens if $U(1)$ symmetry is tacked on as well. This example is easy, since $BU(1) = CP^\infty$, and since complex projective space has one cell in each even dimension, the homology and cohomology are easy to work out explicitly. We get

$$H^n(U(1); \mathbb{Z}) = \mathbb{Z}^{[n+1]/2}. \quad (20)$$
That is, $H^n(U(1); \mathbb{Z})$ is trivial if $n$ is odd and is $\mathbb{Z}$ if $n$ is even. Since elements in $H^*(U(1); \mathbb{Z})$ are always either free or trivial, the torsion part in the Kunneth exact sequence vanishes, and we get an isomorphism

$$H^2(G \times U(1); A) \cong H^2(G; A) \oplus H^0(G; A).$$  \hspace{1cm} (21)

For general actions of $G$ on $A$ we have $H^0(G; A) = A^G$, and so if the action is trivial we get the simple result $H^2(G \times U(1); A) \cong H^2(G; A) \oplus A$. This sort of structure is also expected from the connectedness of $U(1)$.

Finally, in some circumstances it might be useful to get the second group cohomology for $G \times \mathbb{Z}_m$. We get the exact sequence

$$0 \rightarrow H^2(G; A) \oplus (\mathbb{Z}_m \otimes H^0(G; A)) \rightarrow H^2(G \times \mathbb{Z}_m; A) \rightarrow \text{Tor}(\mathbb{Z}_m, H^1(G; A)) \rightarrow 0,$$

and so (for trivial $G$ action on $A$)

$$H^2(G \times \mathbb{Z}_m; A) \cong H^2(G; A) \oplus (A/mA) \oplus mH^1(G; A).$$  \hspace{1cm} (23)

In particular, for $G = \mathbb{Z}_n$ we get the aesthetic $H^2(\mathbb{Z}_n \times \mathbb{Z}_m; A) \cong A_n \oplus A_m \oplus \gcd(n,m)A$.

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