Onsite symmetries for models with arbitrary valency – Part II

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In this short note, we prove that if a string-net phase realizes onsite symmetries (of the kind discussed in previous notes) whose eigenvalues are not $\pm 1$, then the output theory does not admit a braiding. This suggests that looking at the input data from a $G$-graded perspective is the correct way to think about things.

There’s one important aspect of the examples we’ve presented (namely variants on the twisted quantum double) that we haven’t mentioned yet – they don’t admit a braiding! Indeed, no solutions to the hexagon equations exist in any twisted quantum double model except the one with trivial $F$-symbols. I initially discovered this numerically, but it turns out that there’s an easy way to see this in complete generality. This result will prompt us to interpret onsite symmetries in terms of $G$-crossed categories, which will be discussed later.

The main result of this section is as follows:

If $\alpha^n(X) \notin \mathbb{R}$ for some $X \in \mathcal{C}$, then $\mathcal{C}$ does not admit a braiding. \hspace{1cm} (1)

We can prove this by looking at the reverse category $\mathcal{C}_{rev}$, which is defined by taking $\mathcal{C}$ and reversing the tensor product: $X \otimes_{rev} Y = Y \otimes X$. Clearly $\mathcal{C}$ and $\mathcal{C}_{rev}$ are monoidally equivalent (see Etingof for the definition). We are going to show quite generally that if $\mathcal{C}$ and $\mathcal{D}$ are monoidally equivalent, then the onsite symmetries are identical for each of their objects.

Indeed, let $(F, J), F : \mathcal{C} \to \mathcal{D}$ be a functor and natural isomorphism exhibiting the monoidal equivalence, where $J$ establishes the tensor structure of $F$ via

$$F(X \otimes Y) \xrightarrow{J_{X,Y}} F(X) \otimes F(Y). \hspace{1cm} (2)$$

Recall from the last set of notes that we defined

$$\alpha^n(X) = \text{Tr} \left( T^n_X \right) \hspace{1cm} (3)$$

where

$$T^n_X : \text{hom}_\mathcal{C}(1, X^{\otimes n}) \to \text{hom}_\mathcal{C}(1, X^{\otimes n}) \hspace{1cm} (4)$$

is the map that rotates all the legs of the $X^{\otimes n}$ splitting space around by $2\pi/n$. The proof comes from realizing that we can apply the onsite symmetry either before or after acting with the monoidal equivalence, and so the following diagram commutes: (writing $\text{hom}_\mathcal{C}(X, Y)$ as $\mathcal{C}(X, Y)$ for brevity)

$$\begin{align*}
\mathcal{C}(1, X^{\otimes n}) & \xrightarrow{T^n_X} \mathcal{C}(1, X^{\otimes n}) \\
\downarrow F & \quad \quad \quad \quad \quad \downarrow F \\
\mathcal{D}(1, F(X^{\otimes n})) & \xrightarrow{T^n_{F(X)}} \mathcal{D}(1, F(X^{\otimes n})) \\
\downarrow D(1, J^n) & \quad \quad \quad \quad \downarrow D(1, J^n) \\
\mathcal{D}(1, (F(X)^{\otimes n}) & \xrightarrow{T^n_{F(X)}} \mathcal{D}(1, (F(X)^{\otimes n}) \hspace{1cm} (5)
\end{align*}$$

where by $J^n$ has a natural nested structure like the nested associator:

$$J^n = (J^{n-1} \otimes \text{id}) \circ J_{X, X^{\otimes (n-1)}} \hspace{1cm} (6)$$

Now tracing a map like $T^n_X$ gives the same answer regardless of whether we do it before or after mapping with the functor $F$, and so we see that

$$\text{Tr} \left( T^n_X \right) = \text{Tr} \left( T^n_{F(X)} \right) \hspace{1cm} (7)$$
and since $\mathcal{C}$ and $\mathcal{C}_{\text{rev}}$ are monoidally equivalent, we have

$$\alpha^n(X) = \alpha^n(F(X)), \quad \forall \ X \in \mathcal{C}.$$  \hfill (8)

In particular, we have $\alpha^n(X) = \alpha^n(X_{\text{rev}})$.

We’re halfway done. The next thing to do is to show that in fact reversing the braiding conjugates the onsite symmetries – i.e, that $\alpha^n(X) = (\alpha^n(X_{\text{rev}}))^*$ (although perhaps this is intuitively obvious). Indeed, if $F: \mathcal{C} \to \mathcal{C}_{\text{rev}}$ is the functor exhibiting the monoidal equivalence of $\mathcal{C}$ and $\mathcal{C}_{\text{rev}}$, then $T^n_F(X)$ acts on $\mathcal{C}_{\text{rev}}$ by spinning around one $X$ leg clockwise, and performing a nested association on the remaining $X \otimes (n-1)$ bouquet in reverse order compared to the regular nested association in $\mathcal{C}$. This process is precisely the inverse of the process described by $T^n_X$, and so $\text{Tr}(T^n_F(X)) = \text{Tr}(T^n_X)^{-1}$. Since both are roots of unity, we see that $\text{Tr}(T^n_F(X)) = \text{Tr}(T^n_X)^*$.

A more explicit way to see that $\alpha^n(X) = (\alpha^n(X_{\text{rev}}))^*$ for twisted quantum double theories is by looking at the associator in $\mathcal{C}_{\text{rev}}$. Of course, it maps

$$a_{XYZ}^{\text{rev}}: (X \otimes Y \otimes Z) \to X \otimes (Y \otimes Z),$$

but graphically it looks reversed compared to the regular associator:

$$a_{XYZ}^{\text{rev}}: \quad \rightarrow$$

Comparing this to the regular associator in $\mathcal{C}$, we see that

$$a_{XYZ}^{\text{rev}} = a_{XYZ}^{-1}$$

which means that for input categories based on some finite group $G$,

$$\alpha^n_{\text{rev}}(g) = \delta_{g^n, 1} \prod_{k=1}^{n-1} \omega^{-1}(g, g^k, g) = (\alpha^n(g))^{-1} = (\alpha^n(g))^*$$

Collecting our results so far, we see that if $\mathcal{C}$ is braided, then

$$\alpha^n(X) = \alpha^n(X_{\text{rev}}) = (\alpha^n(X))^*, \quad \forall \ X \in \mathcal{C}$$

So we see that if $\mathcal{C}$ is braided then the eigenvalues of every onsite symmetry must be real.

The main point of this note has been to show that the way to think about onsite symmetries should really be in terms of $G$-graded fusion categories, since phases that have nontrivial onsite symmetries still satisfy the ’heptagon’ equations for braiding in $G$-crossed categories. This makes it tempting to regard the onsite symmetries as actual $G$-defects embedded in some background category $\mathcal{C}_0$. How correct this interpretation is remains to be seen, and there are still a few questions I need to think about more (i.e, is it physically sound to gauge the onsite symmetries? How particle-like are they, really?).