Breaking tetrahedral symmetry

Ethan Lake
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In this note, we outline a procedure for realizing topological phases based on string-net models that break tetrahedral symmetry. A few such phases can be realized for abelian groups in the twisted quantum double model, but breaking tetrahedral symmetry in the representation picture is a bit trickier. Besides accessing new topological phases, our approach allows us to determine the relationship between onsite symmetries and input fusion category data. The general procedure is to choose a given set of fusion rules and onsite symmetry transforms and then self-consistently calculate whether or not the chosen set of parameters gives a valid theory. As a check, we are able to reproduce the twisted quantum double models without using cohomology. Additionally, we find that the onsite symmetries under consideration are always trivial for the Fibonacci model, but that a $\mathbb{Z}_3$ onsite symmetry is realized when the input data comes from $\text{Rep}(S_3)$.

I. GENERAL THEORY

The goal is to determine what type of onsite symmetries can exist in general string-net models using the representation picture. In the last note, we saw that nontrivial onsite symmetries were only possible for abelian string-net phases when the input data derived from a “twisted” quantum double – i.e., the integer $p$ parametrizing the $\mathbb{Z}_N$ solutions had to be non-zero (or the matrix $P$ had to be nonsingular in the $\mathbb{Z}_N \times \mathbb{Z}_M$ case). Additionally, in order for the onsite symmetry to be nontrivial, the structure of the symmetry group had to be similar to the structure of the input data.

Unfortunately, the standard Levin-Wen model doesn’t allow us to access phases like this. The reason is because when the $F$-symbols have full tetrahedral symmetry, the choice $p = 0$ (in the twisted quantum double) is forced, and no onsite symmetries can be realized. This motivates us to relax the requirement of tetrahedral symmetry on the $F$-symbols, so that we can investigate the behavior of more general string-net phases.

In this note, we assume our initial data is created from some category $\mathcal{C}$ with multiplicity-free fusion rules: $\dim \text{Hom}(i, j \otimes k) = \delta_{ijk}$. The basis vectors in $\text{Hom}(i, j \otimes k)$ are drawn as trivalent vertices, and labeled by the tensor $T^{jk}_i$:

\[
T^{jk}_i = \begin{array}{c}
  j \\
  i \\
  k
\end{array}
\]  

The basis elements of $\text{Hom}(i \otimes j, k)$ are given by

\[
T^k_{ij} = \begin{array}{c}
  k \\
  i \\
  j
\end{array}
\]

where the counterclockwise indexing is due to the fact that $(i \otimes j \otimes k)^* = k^* \otimes j^* \otimes i^*$. When we “stack” two $T$’s on top of each other, we get something proportional to the identity:

\[
T^{jk}_i T^l_{jk} \propto k
\]

In this note, we’ll freely switch between using “upward flow of time” conventions and explicitly putting directions on the edge labels. We’ll stick to the convention of reading the basis vectors in the flow of time picture clockwise from the top left (which is okay since not all of their edges are oriented upwards).

We draw the $F$-move as

\[
j \quad \quad \quad i \quad \quad \quad m \quad \quad \quad l \rightarrow \sum_n n F^{ijk}_{lnm} i \quad \quad \quad j \quad \quad \quad k \quad \quad \quad n \quad \quad \quad m \]

Which is written algebraically as

\[ T_{ij}^{mn} \triangleq \sum_n (T_{ij}^{nk}T_{nk}^{ml})F_{lm}^{ij} \]  

(5)

To examine how the \( T_{ij}^{jk} \) transform under onsite symmetries, we need to establish some kind of action on them that permutes their indices. We’ll denote the eigenvalue of a vertex under an internal rotation by \( 2\pi/3 \) as \( R_{ijk} \). We see that under a gauge transformation on the rotational degree of freedom, the tensor \( T_{ij}^{jk} \) transforms under rotation as

\[ T_{ij}^{jk} \rightarrow R_{ijk}T_{jki} \]  

(6)

or graphically, as

\[ \begin{array}{ccc} i & j & k \\ \rightarrow & R_{ijk} & \rightarrow \\ j & k & i \end{array} \]  

(7)

If \( j \in L \) is self-dual, define the duality map \( \omega_j \in \text{Hom}(j \otimes j, 1) \) by

\[ w_j = \begin{array}{ccc} j & j \\ \rightarrow & \rightarrow \end{array} \]  

acting on this with duality and using \( \text{dim} \text{Hom}(j \otimes j, 1) = 1 \), we have

\[ \begin{array}{ccc} j & j \\ \rightarrow & \rightarrow \end{array} = \alpha_j \]  

(8)

for some \( \alpha_j \in \pm 1 \), called the Frobenius-Schur indicator. Here we have assumed \textit{rigidity} (valid in any ribbon fusion category), namely that

\[ \begin{array}{ccc} j & j \\ \rightarrow & \rightarrow \end{array} \]  

(10)

If we are working over a ribbon fusion category, we can write \( \alpha_j = \theta_j R_{ij}^{jj} \). (The universal \( R \)-matrix, not the \( R_{ijk} \) used earlier!)

A convenient way of drawing the \( F \)-move is to associate it with a tetrahedron as follows:

\[ F_{lm}^{ij} \sim n \]  

(11)

There are three generators of tetrahedral symmetry – one rotation of order 3 about a face+vertex pair, one rotation of order two connecting two edges, and an inversion about a plane that passes through an edge. The \( F \)-symbols transform as (choosing one possible generating set of symmetries)

\[ \begin{align*}
F_{ln}^{ij} & \rightarrow F_{in}^{jm} & \text{order 3 rotation} \\
F_{ln}^{ij} & \rightarrow F_{nl}^{ki} & \text{order 2 rotation} \\
F_{ln}^{ij} & \rightarrow F_{ln}^{kj} & \text{inversion}
\end{align*} \]  

(12)
We’ll get more explicit about the form of each of these transformations in a second. First, we need some results that will help us in calculations.

When deriving the action of the generators of the tetrahedral group on the input fusion data and doing graph mutations, we will often need to know how to reverse strings. Normally we assume that we can reverse strings and conjugate their labels with impunity – but this is not true when we have onsite symmetries to worry about. As an example, suppose we want to reverse the middle string in the definition of the $F$-symbol. We can accomplish this as follows (schematically, just to give a feel for how the calculations go through):

\[ \sim \rightarrow \sim \rightarrow \sim \rightarrow \sim \rightarrow R \rightarrow \sim \rightarrow \alpha \rightarrow \alpha \rightarrow (13) \]

The $\sim$’s indicate an equality, the $R$ signifies an onsite rotation, and the $\alpha$ means that we pick up a Frobenius-Schur phase factor. This sequence of diagrams has revered (and conjugated) the string in the middle of the $F$-move, and we’ve obtained (in general) a nontrivial phase as a result.

This type of approach in principle allows us to derive how the fusion data transforms under the generators of the tetrahedral group. However, it’s actually easier to express the action of the generators of the tetrahedral group directly in terms of the input fusion data. We can express the action of the tetrahedral group on the fusion data by tensors $F_{ijk}$, defined as

\[ F_{ijk} \equiv F_{ij}^{i'j'k'} ; \]

(14)

We will need two useful identities involving $F$. One is that it can be inverted to “undo” $F$-moves:

\[ i \frac{j}{k} = \frac{1}{F_{ijk}} i \frac{j}{k} \]

(15)

Also, we can act on basis elements in $\text{Hom}(i \otimes j, k)$ (instead of those in $\text{Hom}(i, j \otimes k)$) with $F$ as follows:

\[ = F_{ijk} \]

(16)

Technically, there will be prefactors in front of the $F$ for a general gauge choice of the input fusion data. However, we will always choose to work in a gauge where these factors are one (more on this later). Now it’s time to actually look at the different generators of tetrahedral symmetry. The order 3 rotation corresponds to

\[ F_{ijk}^{l;nm} \sim n \frac{l}{m} \frac{k}{i} \rightarrow F_{ij}^{m*in} \sim l \frac{n}{m} \frac{k}{i} \rightarrow T_{in}^{l} T_{i}^{j} T_{j}^{m} T_{m}^{*} T_{k}^{*} \]

(17)

Likewise, the order 2 rotation corresponds to

\[ F_{ijk}^{l;nm} \sim n \frac{l}{m} \frac{k}{i} \rightarrow F_{ij}^{k*lm} \sim n \frac{l}{m} \frac{k}{i} \rightarrow T_{in}^{l} T_{i}^{j} T_{j}^{m} T_{m}^{*} T_{k}^{*} \]

(18)
FIG. 1: Showing how to compute the action of the order 2 rotation.

Finally, the inversion is

\[ F_{ij;nm} \sim k \]

\[ \Rightarrow \quad F^{m*}_{ij;n} \sim n^* \]

\[ \Rightarrow \quad T^{n^*}_{i^*k^*} T^{m*l^*} T^{m*l^*} T^{ji}_{m^*} \]  \hspace{1cm} (19)

When we derived the relation between onsite rotations and the input data (specifically, the \( F \) tensors) in the last
note, we moved the dots around explicitly. Here we will show that we get the same result by deforming the graph, rather than moving the dots. The transformation of the tensor $T^{ijk}$ under the action of the onsite rotation is

$$T^{ijk} \sim \begin{array}{c} \alpha_k \rightarrow \end{array} T^{jki}$$

which verifies that

$$R^{ijk} = \alpha_k F^{ijk}. \quad (21)$$

We note that this is more general than the $R^{ijk} = \alpha_k$ assumed by Yuting in his work.

We need to compute a way for finding the phase factors generated under the action of the tetrahedral group on the input fusion data. As stated earlier, we can do this entirely through the $F$ tensors. We need to do this for each generator, but I won’t explicitly include the derivation for each identity here. However, I will show the derivation for the order 2 rotation as an example. The derivation on the scanned page shows us that the input fusion data transforms as

$$F_{t_{lmn}} = F_{jm}^{jm} F_{jm}^{jm} F_{jm}^{ml} = F_{in}^{il} F_{in}^{in} F_{in}^{kn} = \alpha_l F_{ijm} \quad (22)$$

One natural condition we might set on the $F$ tensors is

$$F^{ijk} F^{kij} F^{kij} = 1 \quad (23)$$

but this could conceivably be too restrictive – more generally, for labels $i,j,k,l,n,m$ with $F_{t_{lmn}}^{ijk} \neq 0$ we require that a $2\pi$ rotation of the tetrahedron associated with $F_{t_{lmn}}^{ijk}$ gives a factor of 1, i.e. that

$$F_{jm}^{jm} F_{jm}^{jm} F_{jm}^{ml} = F_{in}^{il} F_{in}^{in} F_{in}^{kn} \quad (24)$$

Of course, the $F^{ijk}$ must also satisfy the following pentagon equations:

$$\sum_n F_{t_{lmn}}^{ijm} F_{t_{lmn}}^{kl} = F_{t_{lmn}}^{ijm} F_{t_{lmn}}^{kl} = F_{t_{lmn}}^{ijm} F_{t_{lmn}}^{kl} \quad (25)$$

We note that the onsite symmetry parameters are related directly to the $F$ tensors, and that the $F$ tensors determine how tetrahedral symmetry is broken. Therefore, we can take two possible steps forward. One is to assume a given set of input data and $F$ tensors (hence assuming a given form of tetrahedral symmetry breaking) and then derive the relations for the onsite symmetries. The other is to assume a set of onsite symmetry parameters and fusion rules and then to see if a set of input data exists which realizes the chosen parameters. The second approach has seemed easier so far.

II. THE DETAILS

As in the last note, the $R$-tensors, FS indicators, and quantum dimensions all have a built-in gauge degree of freedom – indeed, if $g : L \rightarrow U(1)$ is any homomorphism and $\{R^{ijk}, \alpha, d_i\}$ is a valid set of data, then so too is $\{R^{ijk} g(k^*) \}$, $g(i)$, $g(k)$, $g(i)$ $g(k)$. Additionally the $F$-symbols, being related to cocycles, have a gauge degree of freedom built in to them. They transform under the action of $f : L \times L \times L \rightarrow C$ as

$$f_{t_{lmn}}^{ijk} \rightarrow F_{t_{lmn}}^{ijk} F_{t_{lmn}}^{ijk} \quad (26)$$

We find it extremely helpful if we gauge the $F$-symbols so that they are 1 if any of the top lines $(i,j)$ or $k$ is the identity. If this condition is satisfied, we will call the resulting fusion data homogeneous. Homogeneity is accomplished if

$$f_{i_0}^0 = f_{ij}^i = f_{00}^0 = (f_{ij}^i)^{-1} = (f_{0i}^i)^{-1} = (f_{0}^0)^{-1} \quad (27)$$
We will usually take $f_i^0 = f_i^0 = 1 \forall i \in L$. Without loss of generality, we can choose to work in a unitary gauge where $F_{i^0}^{i^0} = F_{i^0}^{i^0}$. This ensures that the prefactors in front of the transformations governed by the $F$ tensors are all unity. The quantum dimensions are defined through the relation

$$d_i = \frac{1}{\sqrt{F_{i^1}^{i^0} \sqrt{F_{i^0}^{i^1}}} (28)}$$

which is invariant under $f$ gauge transformations. In general, the fusion systems we can construct that satisfy the above conditions will be in $1-1$ correspondence with all possible fusion categories (up to $\mathbb{C}$-linear monodial equivalence).

As mentioned earlier, we assume a given set of fusion rules and onsite symmetries, and then see if any fusion category exists that realizes the chosen parameters. To do this, we use the following algorithm:

- Choose a set of fusion rules and determine the irreducible representations of the group that the rules are derived from.
- Calculate the $3j$-symbols for the theory, imposing the cyclic symmetry condition.
- Calculate the $6j$-symbols for the theory, assuming full tetrahedral symmetry.
- Choose an ansatz for the onsite symmetry operators.
- Homogenize the $6j$-symbols by an appropriate $f$ gauge transformation.
- Partition the $6j$-symbols into orbits based on their transformation under the generators of the tetrahedral symmetry group.
- Solve the pentagon identity self-consistently on the representative elements of each orbit of the input data under the tetrahedral symmetry group to determine whether the initial onsite symmetry ansatz is allowable.

The last step is the most difficult. We only have to solve the pentagon equations after modding out by the tetrahedral symmetry action, but this is still a difficult task in general. I worked through the Fibonacci example by hand, but anything more complex (like $S_3$) I’ve had to handle with a computer.

### III. EXAMPLES

The $6j$ fusion systems for finite abelian groups $G$ are in $1-1$ correspondence with the cohomology group $H^3(G; \mathbb{C}^\times)$. Thus the number of fusion categories is the same as the number of orbits of $H^3(G; \mathbb{C}^\times)$ under actions of $Aut(G)$. As we saw in the last note, if $G = \mathbb{Z}_n$, then $H^3(G; \mathbb{C}^\times) \cong \mathbb{Z}_n$, and we can generate the $6j$-symbols by the 3-cocycle $f : G \times G \times G \to U(1)$ given by

$$f(k, l, m) = \exp\left(\frac{2\pi ik}{N^2} (l + m - [l + m] z_N)\right). \quad (29)$$

which is known as the twisted quantum double.

However, the goal is to encompass the twisted quantum double model in the standard representation-theory picture, and so we will avoid using cohomology for now. The point here is to check to see whether our algorithm for realizing more general phases can work in the simple case when the input data is based on finite abelian groups.

#### A. $\mathbb{Z}_2$

If the label set is $L = \mathbb{Z}_2$, then every $R$ tensor is trivial, due to our homogeneous gauge condition on the $F$-symbols and the fact that when working with abelian groups if $l = 0$ then we must have $\delta_{ijk} = 1$. There are then only two solutions to (25), which reads $(F_{1^0}^{1^0})^2 = 1$, and so $F_{1^0}^{1^0} = \pm 1$ (implying $d_0 = 1$, $d_1 = \pm 1$). These are of course the toric code and doubled semion models, respectively. Normally we get the doubled semion theory with the deformation parameter $q = \exp(\pi i/3)$. By contrast, here we don’t have to think about quantum groups at all!
The $R$-tensors we need to determine are $R^{111}$ and $R^{222}$. The choice $R^{111} = R^{222} = 1$ gives the standard tetrahedrally symmetric $Z_3$ theory. The quantity $R^{111}R^{222}$ is gauge-invariant, since it transforms under an $f$ gauge transformation as

$$R^{111}R^{222} \rightarrow \frac{f_{11}^{12} f_{10}^{21} f_{12}^{01}}{f_{12}^{10} f_{10}^{12} f_{12}^{21}} R^{111}R^{222} = R^{111}R^{222}$$

(30)

We want to realize an onsite $Z_3$ symmetry, and so we choose $R^{111} = \omega$, $R^{222} = \bar{\omega}$, where $\omega$ is a third root of unity. We also make the choice that $\alpha_1 = \omega = \bar{\alpha}_2$. This choice ensures that we begin with trivial quantum dimensions for each string label. A subsequent $g$ gauge transformation with $g(j) = \exp(-2\pi ij/3)$ kills the FS indicators, and leaves us with the data

$$d_j = \omega^j p^p, \ R^{222} = \omega^{-p}$$

(31)

where $p = \pm 1$.

We now try do determine whether this model can be realized by some tetrahedral-symmetry-breaking input fusion data in the representation picture. Our original choice of FS indicators ensures that

$$F_{1,00}^{121} = \omega^p, \ F_{2,00}^{212} = \bar{\omega}^p$$

(32)

while our choice of $R$-tensors implies

$$F_{0,22}^{111} = 1, \ F_{0,12}^{222} = \bar{\omega}$$

(33)

The fusion rules tell us that the only other possible nontrivial $F$-symbols are $F_{1,00}^{112}, \ F_{2,00}^{122}, \ F_{2,00}^{212}, \ F_{2,00}^{221}, \ F_{2,00}^{121}$. They can all be determined from our earlier relation for the order 2 rotation of the tetrahedron, our expression for the order 3 rotation, and the pentagon identity. Making use of our expressions for the action of the tetrahedral group cuts down on the number of variables to solve for, and the final system of equations can easily be solved by hand. Up to gauge transformations, we find only one solution:

$$F_{1,02}^{112} = F_{1,00}^{121} = F_{2,10}^{122} = \omega^{2p}, \ F_{2,00}^{212} = F_{2,01}^{221} = F_{0,11}^{222} = \omega^p, \ \text{others} = 1$$

(34)

which is in agreement to the input data provided by the twisted quantum double of $Z_3$. Note however that we haven’t used any cohomology to get here!

C. $S_3$

We want to see if we can realize onsite symmetries in a non-abelian group. The simplest ones are the odd dihedral groups, since we can compare our results to what we know from cohomology. First, we work out $S_3$ from a cohomological point of view so that we have something to compare our representation picture results with.

The most convenient way to label the group elements in the odd dihedral groups is as pairs $(a, \alpha)$, where $a \in \mathbb{Z}_n$ and $\alpha \in \mathbb{Z}_2$ (which is possible due to the relation $D_n \cong \mathbb{Z}_2 \times \mathbb{Z}_n$). Group multiplication is done as

$$(a, \alpha) \cdot (b, \beta) = (\{(-1)^\beta a + b\}_{2^{n+1}}, \{\alpha + \beta\}_{2^2}).$$

which means that we invert elements by

$$(a, \alpha)^* = ((-1)^{\alpha+1} a, \alpha).$$

The third cohomology group decomposes as

$$H^3(D_{2n+1}, U(1)) \cong \mathbb{Z}_{2n+1} \times \mathbb{Z}_2 \cong \mathbb{Z}_{4n+2}$$

We again parametrize the different solutions by $p$, where $p \in \mathbb{Z}_{2n+1}$. The 3-cocycles are

$$F((a, \alpha), (b, \beta), (c, \gamma)) = \exp\left(\frac{2\pi ip}{(2n+1)^2} \left((-1)^{\beta+\gamma} a ((-1)^\gamma b + c - ((-1)^\gamma b + c)_{2^{n+1}}) + \alpha \beta \gamma \frac{(2n + 1)^2}{2}\right)\right).$$
Of course, we want to focus on $D_3$. I won’t write out the $F$-symbols here, for brevity’s sake (they’re even worse than $\mathbb{Z}_6$, due to the two-component notation we’re using!) Before doing any gauge transformations, we find that the parity factors are

$$\alpha_{01} = -1, \quad \alpha_{10} = \omega, \quad \alpha_{11} = -1, \quad \alpha_{20} = \omega^2, \quad \alpha_{21} = -1, \quad \text{others} = 1 \quad (35)$$

We can then do an $f$-gauge transformation with $f(20, 10) = \omega$ and $f(10, 20) = \omega^2$, followed by a $g$ transformation with $g(x) = \text{sgn}(x)$, where $\text{sgn}(x)$ is the parity of $x$. This kills off the onsite parity symmetry. In fact, this sort of gauge transformation is possible for all odd dihedral groups.

However, the onsite rotational symmetry cannot be gauged away. In particular, $R_{aa} R_{a^*a^*}$ is still a gauge-invariant quantity. In the case of $D_3$, we find $R_{10,10} R_{20,20} = \omega^2$. Additionally, the quantum dimensions are changed to $d_x = \text{sgn}(x)$ (EL: is this interesting?). For general odd dihedral groups with $3(2n + 1)$, we find that the rotation is always nontrivial for all $p \in \mathbb{Z}_{2n+1}^\times$, and that if the rotational symmetry is nontrivial, then the quantum dimensions satisfy $d_x = \text{sgn}(x)$.

So we see that in the group element picture, $S_3$ has a nontrivial onsite $\mathbb{Z}_3$ symmetry. Does this hold in the representation picture as well? The number of label types is only 3 in the representation picture, and they are all self-dual. The fusion rules are drastically different in each case and the number of $F$-symbols we need to compute in the representation picture is a bit larger, and so it is by no means obvious if the $\text{Rep}(S_3)$ model will realize any symmetries.

We let the group elements be denoted as $\{0, 1, 2\}$, with 1 the sign representation and 2 the 2-dimensional irrep. We first go though and obtain the tetrahedrally-symmetric $F$-symbols via the usual procedure. We choose an $f$-gauge transformation where $f^{00}_0 = 1$ and $f^{ij}_k = (\sqrt{i + j + k})/2$ as long as one of $i,j,k$ is nonzero. This ensures that the resulting $F$-symbols are homogeneous.

We then choose the ansatz $R^{222} = \omega$, with all other $R$ trivial. We also assume trivial FS indicators. This ansatz gives us the complete set of $F$ tensors, and we use them to partition the input data into orbits under the tetrahedral symmetry group. This is useful because we can often use the tetrahedral symmetry action to relate $F$-symbols that we could not initially constrain (like $F^{222}_{220}$) to ones that were set to be trivial through homogeneity. After this is done, we must solve the pentagon equations for the remaining few undetermined $F$-symbols. In this example, we require a computer to complete this step.

We find that our initial ansatz for the onsite symmetry conditions works out – we can find a set of $F$-symbols that satisfy the pentagon equation and where the rotational $\mathbb{Z}_3$ onsite symmetry cannot be gauged away. Of course, the final fusion data also breaks tetrahedral symmetry, and so is inaccessible through the regular Levin-Wen model. Although this is just one example, it seems that the structure of the group the input data is based on determines when the onsite fusion data also breaks tetrahedral symmetry, and so is inaccessible through the regular Levin-Wen model. This seems like a fairly nontrivial check of string-net / toric code duality, although I haven’t been able to prove the sort of general statements here that I was able to make in the abelian group case.

### D. Quantum groups

We now try to determine if we can realize onsite $\mathbb{Z}_3$ or $\mathbb{Z}_2$ symmetries for a theory derived straight from quantum groups. We first consider the Fibonacci theory, using the notation $L = \{0, 1\}$. We choose the gauge condition given by

$$f^{ij}_{k} = (\sqrt{\phi})^{i+j+k} \quad (36)$$

which guarantees that the $F$-tensors are homogeneous:

$$F^{11}_{1,00} = 1/\phi, \quad F^{11}_{1,01} = F^{11}_{1,10} = 1/\sqrt{\phi}, \quad F^{11}_{1,11} = -1/\phi, \quad \text{others} = 1 \quad (37)$$

We also choose the ansatz $R^{111} = \omega$, with $\alpha_0 = 1$ but $\alpha_1$ allowed to float as a free parameter. The Fibonacci theory is much easier than $S_3$, and we can finish up everything by hand. After a bit of work, we find that no nontrivial onsite $\mathbb{Z}_3$ or $\mathbb{Z}_2$ onsite symmetry can be realized – assuming $R^{111} R^{000} \neq 1$ ends up violating the pentagon identity, regardless of $\alpha_1$. This doesn’t tell us much, since we might not have expected an onsite $\mathbb{Z}_3$ symmetry anyway (as the “$\mathbb{Z}_3$”-ness of the Fibonacci theory is pretty unclear).

One might have guessed that the doubled Fibonacci theory might be a good place for realizing the onsite $\mathbb{Z}_2$ symmetry, based on our experience with the $\mathbb{Z}_2 \times \mathbb{Z}_2$ model from the last note. For now, looking at this in depth looks like it will require more work than it’s worth – I’ll revisit it later if needed.

To be continued...