1) a) Relation between $\theta$ & $b$?

\[ b = R \sin \beta \quad \text{&} \quad \beta = \frac{\pi - \theta}{2} = \frac{\pi - \frac{b}{R}}{2} \]

\[ b = R \sin \left( \frac{\pi - \frac{b}{R}}{2} \right) = R \cos \frac{\theta}{2} \cdot \frac{b}{R} \]

b) Cross section for $\theta > \frac{\pi}{2}$

\[ \theta > \frac{\pi}{2} \Rightarrow b < R \cos \frac{\pi}{4} \quad \text{or} \quad b < \frac{R}{\sqrt{2}} \]

& the cross section is $\sigma_{\theta>0} = \frac{\pi R^2}{2}$.

c) Cross section for $\theta > 0$

As soon as $b < R$, $\theta > 0$ so $\sigma_{\theta>0} = \pi R^2$

As soon as $b \geq R \theta = 0$ so $\sigma_{\theta=0} = \infty$.

d) Rutherford cross section for $\theta > 0$ is as comment.

This results from the fact that the electrostatic force has an infinite range. No matter how large $b$, the impact parameter, there is a deflection of the incoming particle. This is quite different from this exercise, where for $b > R$ we have $\theta = 0$. 
2) \( \alpha \) with kinetic energy \( K_\alpha = 7.7 \text{ MeV} \)

a) Upper limit on the Gold nucleus radius.

In precisely head on collision, the \( \alpha \) particle comes to a stop before reaching the nucleus where \( \frac{1}{2} \frac{m_\alpha}{E_\alpha} \frac{Q_\alpha Q_{\text{Gold}}}{R_{\text{min}}} = K_\alpha \) to 20

\[
R_{\text{min}} = \frac{1}{4\pi \varepsilon_0} \frac{Q_\alpha Q_{\text{Gold}}}{K_\alpha} = 9 \times 10^9 \text{ N m}^2 \text{C}^{-2} \frac{4 \times 79 \times (1.6 \times 10^{-19} \text{C})^2}{7.7 \times 10^6 \times 1.6 \times 10^{-19} \text{ N m}}
\]

\( R_{\text{min}} = 5.9 \times 10^{-14} \text{ m} \). This distance of closest approach is an upper limit for the Gold nucleus radius.

b) Gold nucleus mass density.

The mass of a Gold nucleus is \( M_{\text{Au}} = \frac{197}{197} 10^{-3} \text{ kg} \) & its radius is less than \( R_{\text{min}} \) so, the mass density \( \rho > \frac{M_{\text{Au}}}{\frac{4\pi}{3} R_{\text{min}}^3} \)

\[
\rho > \frac{197 \times 10^{-3}}{6.02 \times 10^{23} \times 4 \pi \times (5.9 \times 10^{-14})^3} = 3.8 \times 10^{14} \text{ kg m}^{-3}
\]

c) Size of the Sun with such a density?

\[
M_{\text{Sun}} = \frac{4\pi}{3} R_{\text{Sun}}^3 \times \rho \Rightarrow R_{\text{Sun}} = \left( \frac{3 M_{\text{Sun}}}{4\pi \rho} \right)^{1/3}
\]

\[
R_{\text{Sun}} = \left( \frac{3 \times 2 \times 10^{30} \text{ kg}}{4\pi \times 3.8 \times 10^{14} \text{ kg m}^{-3}} \right)^{1/3} = 1.08 \times 10^5 \text{ m} = 1.08 \text{ km}
\]

If the Sun (whose diameter is ~100 times that of the Earth) has the same mass but the density of nuclear material, its radius would be less than 100 km!
3) Rotation level of the hydrogen molecule
   
a) Moment of inertia
   The hydrogen nucleus is 2000 times more massive than an electron & has a radius of \(\approx 10^{-15}\) m, negligible when compared to the size of the molecule \(\Rightarrow\) Hydrogen atoms in the molecule can be treated as point-like objects.
   
The moment of inertia is \(I = 2 M(R)^2 = \frac{MR^2}{2}\).

b) Quantization of energy.
   The kinetic energy is \(K = \frac{1}{2} I \omega^2\) & the angular momentum is \(L = I \omega\) so \(K = \frac{1}{2} I \frac{L^2}{I^2} = \frac{L^2}{2I}\).
   
   Quantizing the angular momentum \(L = n \hbar\) with \(n = 2, 3, \ldots\).
   
   (In fact, we will soon see that the proper quantization is \(L^2 = n(n+1) \hbar^2\) with \(n = 0, 1, 2, \ldots\).
   
   So \(K_n = \frac{1}{2} \frac{2n^2 \hbar^2}{MR^2} = \frac{n^2 \hbar^2}{MR^2}\) (in fact \(n(n+1) \hbar^2\)).

c) Wavelength of the lowest energy transition?
   Transition between levels \(n \leftrightarrow m\) will have energies \(\hbar^2 (n^2 - m^2) / MR^2\)
   with \(n > 0 \& m > 0 \& n \neq m\) (In fact \(n=0 \& m=0\) are ok).
   
   The smallest transition energy is between \(n=1 \leftrightarrow m=2\) so
   \[E_{11} = \frac{(4-1) \hbar^2}{MR^2} = \frac{\hbar^2 c}{\lambda} = \frac{2\pi \hbar}{\lambda} \] \[\Rightarrow \lambda_{11} = \frac{2\pi MR^2}{(4-1) \hbar} = 25 \mu m\.
   
   Note: In fact \(E_{nm} = \frac{\hbar^2}{MR^2} (n(n+1) - m(m+1)) \& \lambda_{nm} = \frac{2\pi MR^2}{\hbar} \frac{1}{n(n+1) - m(m+1)}\)
   
   The transition with the smallest energy (and the largest wave length) is
   \[\lambda_{10} = \frac{2\pi MR^2}{\hbar} = 37.4 \mu m\]