1) **Space time diagrams**

   $A$ and $B$ devices separated by a distance $d = 20\text{m}$ measured in the reference frame $S$ where they are at rest. The following sequence of events occurs:

   (K) $A$ sends a light signal towards $B$.
   (L) As soon as the signal is received by $B$, $B$ sends a signal back to $A$.
   (M) $A$ receives the signal from $B$.

   The reference frame $S'$ coincides with $S$ at time $t = t' = 0$ and is in motion with respect to $S$ at speed $\beta = v/c = 0.6$ in the direction of increasing $x$.

   a) (10 points) Draw the space time diagram $(ct, x)$ showing events K, L and M from the point of view of the reference frame $S$. Indicate slopes and $ct$ intervals between events on your diagram.

   World lines of light signals

   have slopes of $\pm 1$.

   ![Diagram](image)

   b) (10 points) From the point of view of $S'$, what is the distance $d'$ separating $A$ and $B$?

   For $S'$, the $AB$ distance is affected by length contraction $d' = \frac{d}{\gamma}$

   with $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(0.6)^2}} = \frac{1}{0.8} = \frac{5}{4} \Rightarrow d' = 20\text{m} \times \frac{4}{5} = 16\text{m}

   $d' = 16\text{m}$

   c) (10 points) From the point of view of $S'$, what is the time interval $c\Delta t_{KM}'$ between event (K) and event (M)?

   The time interval between $K$ & $M$ is affected by time dilation $c\Delta t = \gamma\Delta t$

   $c\Delta t' = \frac{5}{4} \times 20\text{m} = 50\text{m}

   $c\Delta t' = 50\text{m}$
e) (10 points) Using a Lorentz transform, find the time interval $c \Delta t_{KL}'$ between K and L and $c \Delta t_{LM}'$ between L and M.

\[
K = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad L = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} 40 \\ 0 \\ 0 \end{pmatrix}
\]

\[
\gamma = \frac{5}{4}, \quad \beta = \frac{6}{10} = \frac{3}{5}
\]

The Lorentz transformation giving events coordinates in $S'$ form

\[
K' = \Lambda K = \begin{pmatrix} 0 \\ 0 \\ 50 \end{pmatrix}, \quad L' = \Lambda L = \begin{pmatrix} 10 \\ 0 \\ 10 \end{pmatrix}
\]

& \quad M' = \Lambda M = \begin{pmatrix} -30 \\ 0 \\ 50 \end{pmatrix}

The time difference between K & L is $c \Delta t_{KL}' = 10 \, m$

The time difference between L & M is $c \Delta t_{LM}' = 50 - 10 = 40 \, m$

\[
\begin{array}{l}
\text{c} \Delta t_{KL}' = 10 \, m \quad \text{&} \\
\text{c} \Delta t_{LM}' = 40 \, m
\end{array}
\]

b) (10 points) Draw the space time diagram $(ct', x')$ from the point of view of reference frame $S'$ with the events K, L and M, summarizing the results of your above calculations.

\* A & B are 16m apart
\* The world lines of A & B have a slope of $\frac{1}{\beta} = \frac{1}{0.6} = \frac{5}{3}$
2) Here comes the muon again
A muon $\mu$ with mass $m_\mu = 105.5\text{MeV}/c^2$ has a kinetic energy $K_\mu = 211\text{MeV}$ measured in the lab reference frame.

a) (8 points) What is the total energy $E$ of the muon $\mu$?

$$E = m_\mu c^2 + K_\mu = 105.5 + 211 = 316.5\text{MeV}$$

b) (8 points) What is the momentum $p$ of the muon $\mu$?

$$p = \sqrt{E^2 - m_\mu^2 c^4} = \sqrt{(316.5)^2 - (105.5)^2} = 298.4\text{MeV/c}$$

$$c) (8 \text{ points}) \text{ What is the speed } u \text{ (in terms of the speed of light) of the muon } \mu?$$

$$u = \frac{p}{E} = \frac{298.4}{316.5} = 0.9428$$
We now consider the same muon $\mu$ but this time from the point of view of a reference frame moving in opposite direction a speed $v = 0.8c$ with respect to the lab.

d) (18 points) What are the total energy $E'$ and momentum $p'$ of the muon $\mu$?

\[
\begin{align*}
\mathbf{p} &= \begin{pmatrix} E \\ -pc \end{pmatrix} \\
\mathbf{p}' &= \begin{pmatrix} \gamma & -\beta y \\ -\beta y & \gamma \end{pmatrix} \begin{pmatrix} E \\ -pc \end{pmatrix} = \begin{pmatrix} E' \\ p'c \end{pmatrix}
\end{align*}
\]

\[
\begin{align*}
\beta = 0.8 & \quad m \gamma = \frac{1}{0.6} = \frac{5}{3} \quad & \beta y = \frac{8}{10} \cdot \frac{5}{3} = \frac{4}{3} \\
E' &= \frac{5}{3} \times 316.5 + \frac{4}{3} \times 298.4 = 925.37 \text{ MeV} \\
p'c &= -\frac{4}{3} \cdot 316.5 - \frac{5}{3} \times 298.4 = -919.33 \text{ MeV} \\
\end{align*}
\]

\[
\begin{align*}
\sqrt{E'^2 - p'^2c^2} &= m^2 \gamma c^2 \\
E' &= 925.37 \text{ MeV} \quad & p' = 919.33 \text{ MeV/c}
\end{align*}
\]

& one can verify that $E'^2 - p'^2c^2 = m^2 \gamma c^4$.

f) (8 points) What is the speed $u'$ (in terms of the speed of light) of the muon $\mu$?

\[
\beta' = \frac{p'c}{E'} = \frac{919.33}{925.37} = 0.99347.
\]

\[
\beta' = 0.99347.
\]