A) Non conservation of classical momentum

- Consider two observers in two reference frames moving at speed $v$ along the $x$ axis.
- Each throws a ball of mass $m$ in the $y$ direction so they collide & bounce back in the $y$ direction.
- If each observer threw the ball with a speed $u$, each should see the ball bounce back with a speed $u$.

- For the observer in $S'$ the ball sent by $S$ has a $y$ velocity of 
  $$ \pm u \sqrt{1 - \frac{v^2}{c^2}} $$
- As seen from $S$, the momentum prior to collision is
  $$ P_1 = m \, u - m \, u \sqrt{1 - \frac{v^2}{c^2}} = m \, u \left( \frac{1 - \sqrt{1 - \frac{v^2}{c^2}}}{c^2} \right) $$
  
  while after the collision the total momentum is
  $$ P_2 = -m \, u + m \, u \sqrt{1 - \frac{v^2}{c^2}} = m \, u \left( \sqrt{1 - \frac{v^2}{c^2}} - 1 \right) $$
  
  so $P_1 \neq P_2$ and momentum is not conserved?!?

- There are two things we can do:
  - We could abandon conservation of momentum.
  - We could redefine momentum.
B) Definitions: Scalars & 4-Vectors

a) Scalar

A scalar is a number, a physical quantity that is the same for all observers, independently from their reference system & state of motion.

Example of scalars:
- Space time intervals
- The proper time between two events
- The electric charge
- The number of apples on a tree.

Example of non-scalar quantities:
- The length of an object
- The time between two events

b) 4-Vectors

Two events separated by $\Delta t$, $\Delta x$, $\Delta y$, $\Delta z$ constitute a four-vector.

It is independent from the observer's reference system in the sense it connects two events happening in all reference systems. This is not saying the components $\Delta t$, $\Delta x$, $\Delta y$, $\Delta z$ of the four vector are the same for all reference systems.
Exactly in the same way, the vector connecting two points in space is independent from the coordinate system used but its components do depend on the coordinate system.

* The components of a 4-vector are transformed from one reference system to another by the means of a Lorentz transform (plus possibly notations which with the Lorentz transforms constitute the Poincaré group).

* Two 4-vectors can be used in a dot product:
  \[ a \cdot b = a_t b_t - a_x b_x - a_y b_y - a_z b_z \]
  
  \[ a \cdot b \] is a scalar.

* A 4-vector multiplied by a scalar is a 4-vector.
  A 4-vector multiplied by a non-scalar is not a 4-vector.

\( \text{c) Four-velocity} \)

* The usual velocity is defined as \( v = \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t} = \frac{dr}{dt} \).

We are looking for a four-vector — with \( \Delta x \) a step in space time described by a four-vector. It is tempting to define \( u = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \) however this is not a four-vector since \( \Delta t \) is not a scalar.
We could however use \( \mu = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} \) with \( \Delta t \)
the proper time
\[
\mu = \frac{dx}{dt} = \left( c \frac{dt}{dt}, c \frac{dx}{dt}, c \frac{dy}{dt}, c \frac{dz}{dt} \right) \quad \& \quad \text{with} \quad \Delta t = \frac{dt}{\gamma} \\
= \left( c \gamma \frac{dt}{dt}, c \gamma \frac{dx}{dt}, c \gamma \frac{dy}{dt}, c \gamma \frac{dz}{dt} \right) \quad \text{with} \quad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \\
= \left( \gamma c, \gamma \nu_x, \gamma \nu_y, \gamma \nu_z \right) = \left( \nu', \nu \right)
\]
Let's check we recover the correct velocity transformation by applying a Lorentz transformation to the 4 velocity:
\[
\beta_0 = \frac{v}{c} \quad \& \quad \gamma_0 = \frac{1}{\sqrt{1 + \beta_0^2}}
\]

\[
\begin{bmatrix}
\gamma_0 & -\beta_0 \gamma_0 & 0 & 0 \\
-\beta_0 \gamma_0 & \gamma_0 & 0 & 0 \\
0 & 0 & \gamma & 0 \\
0 & 0 & 0 & \gamma
\end{bmatrix}
\begin{bmatrix}
\nu' \\
\nu'_x \\
\nu'_y \\
\nu'_z
\end{bmatrix}
= \begin{bmatrix}
\gamma_0 \nu_0 - \beta_0 \gamma_0 \nu_{0x} \\
\gamma_0 \nu_{0x} + \beta_0 \gamma_0 \nu_0 - \nu_{0x} \\
\gamma \nu_{0y} \\
\gamma \nu_{0z}
\end{bmatrix}
= \begin{bmatrix}
\nu' \\
\nu'_x \\
\nu'_y \\
\nu'_z
\end{bmatrix}
\]
\[
\gamma' = \gamma_0 \gamma \left( 1 - \frac{\beta_0 \nu_x c}{c} \right)
\]
\[
\nu'_x = \frac{\gamma_0 \gamma \left( \nu_x - \nu \right)}{\gamma_0 \gamma \left( 1 - \beta_0 \nu_x c \right)} = \frac{\nu_x - \nu}{1 - \nu \nu_x c^2}
\]
\[
\nu'_y = \frac{\gamma \nu_{0y}}{\gamma_0 \gamma \left( 1 - \beta_0 \nu_x c \right)} = \frac{\nu_{0y}}{\gamma \left( 1 - \nu \nu_x c^2 \right)}
\]

You can convince yourself that \( \mu \cdot \mu = c^2 \) by going through a lengthy calculation or, more simply by applying a Lorentz transform to bring you in the reference frame where the particle is at rest. Indeed, there \( \mu' = (c, 0, 0, 0) \) & it is clear then that \( \mu \cdot \mu = c^2 \).
D) Four-Momentum

- You might have been reading about the relativistic mass which increases with the particle velocity - you should just forget about this; when we talk about mass, we mean the mass in the rest frame or such a way the mass is a scalar.

- We define four momentum as \( \mathbf{p} = m \mathbf{u} = (\gamma mc, \gamma m u_x, \gamma m u_y, \gamma m u_z) \).

- The space part of \( \mathbf{p} \) is the new definition of momentum that is uncanceled:
  \[
  \mathbf{p} = \gamma m \mathbf{u} = m \mathbf{u} \left( 1 - \frac{\mathbf{u}^2}{c^2} \right)^{-1/2} \\
  \approx m \mathbf{u} \left( 1 + \frac{1}{2} \frac{\mathbf{u}^2}{c^2} \right) \\
  \approx m \mathbf{u} + \frac{1}{2} m \frac{\mathbf{u}^2}{c^2} \mathbf{u} \quad \text{when} \ \frac{\mathbf{u}}{c} \ll 1 \\
  \mathbf{p} \text{ moves its classical form when} \ \frac{\mathbf{u}}{c} \to 0
  \]

- What about the temporal component?
  \[
  p_0 = \gamma mc = mc \left( 1 - \frac{\mathbf{u}^2}{c^2} \right)^{-1/2} \\
  \approx mc + \frac{1}{2} m \frac{\mathbf{u}^2}{c^2}
  \]
  So \( p_0 c = mc^2 + \frac{1}{2} m \mathbf{u}^2 \) when \( \mathbf{u}/c \ll 1 \).

  It seems that \((\gamma - 1)mc^2\) is the kinetic energy.

  If we consider \( p_0 c = \gamma mc^2 \) as the energy, we see a new term \( mc^2 \) has appeared as an internal energy associated with the mass.

- We have \( E = \gamma mc^2 \) and \( \mathbf{p} = \gamma m \mathbf{u} \).
  \( \mathbf{p} = m \mathbf{u} \) is referred to as the four-momentum or as the energy-momentum.
\[ E = mc^2 \Rightarrow \text{it takes an infinite energy to bring a massive particle to the speed of light.} \]

In the same way a mass less particle to have some energy, can only go the speed of light.

\[ \beta = \frac{u}{c} = \frac{\gamma m v c}{\gamma m c^2} = \frac{pc}{E} \]

\* In the rest frame \( p^2 = m^2c^2 \)

While in the laboratory frame \( p^2 = \gamma m^2 c^2 - \gamma m \dot{\vec{v}}^2 \)

But \( p^2 \) is a scalar so

\[ E^2 = m^2c^4 + \dot{p}^2c^2 \]

\* With this last relation we see again that in the rest frame \( E = mc^2 \). For a rotating particle \( \dot{p} = 0 \) but the rest mass is increased. In the same way, the rest mass of a system of particles is generally smaller than the sum of the rest mass of the constituents - The difference is just the binding energy - This is at the heart of nuclear physics.

\* Photons: \( m = 0 \) implies \( E = pc \) and since \( \beta = \frac{pc}{E} = 1 \)

Massless particles go the speed of light.

A photon going along the x direction has a four momentum \( (p^0, p_x, 0, 0) \) & \( |p|^2 = 0 \) but it still carries energy & momentum.
* Doppler effect revisited.*

- We have seen earlier that if $\lambda'$ is the wavelength emitted in the rest frame, the wavelength in the lab is $\lambda = \gamma (1 + \beta \cos \theta) \lambda'$.
- Using a notation we can bring $\theta$ along the $x$ direction & then apply a Lorentz transform. Photons are massless so $E = pc$.

$$
\begin{pmatrix}
\gamma & -\beta \gamma & 0 & 0 \\
-\beta \gamma & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
E/c \\
-E/c \sin \theta \\
E/c \sin \gamma \\
0
\end{pmatrix}
= 
\begin{pmatrix}
E'/c \\
-E'/c \sin \theta \\
E'/c \sin \gamma \\
0
\end{pmatrix}
$$

We see that $E' = \gamma E (1 + \beta \cos \theta)$.

Soon we will see the energy transferred by a photon is related to its wavelength $E = \frac{hc}{\lambda}$ with

$h = 6.62 \times 10^{-34}$ J.s Planck's constant.

$$\frac{hc}{\lambda'} = \gamma \frac{hc}{\lambda} (1 + \beta \cos \theta) \quad \Rightarrow \quad \lambda = \frac{\gamma (1 + \beta \cos \theta) \lambda'}{\gamma}$$
as we had found earlier.
Problem 10 – Modern Physics

The neutral pion $\pi_0$ has a mass $m_{\pi_0}c^2 = 140 \text{ MeV}$ and decays in a pair of $\gamma$ ray photons with a rest frame mean decay time constant $\tau_{\pi_0} = 10^{-16} \text{ s}$. In the entire exercise, we consider a pion $\pi_0$ with a kinetic energy $K_{\pi_0} = 1.26 \text{ GeV}$ in the laboratory reference frame.

(a) [5 pts.] In the laboratory reference frame, what is the average $\pi_0$ decay time? What is the average distance traveled by the $\pi_0$ in the laboratory reference frame before it decays?

(b) [6 pts.] From first principles, show that, in the reference frame where the $\pi_0$ is at rest, the two $\gamma$ ray photons have the same energy equal to $\frac{1}{2}m_{\pi_0}c^2$.

(c) [7 pts.] What is the energy range within which the photons should be detected in the laboratory reference frame?

(d) [7 pts.] Consider the $\gamma$ ray photons that are emitted in the reference frame where the $\pi_0$ is at rest along a direction making an angle of $\frac{\pi}{2}$ with the direction of motion of the pion $\pi_0$. In the laboratory frame, what is the angle between the direction of these photons and that of the $\pi_0$?
Problem 10 - Modern Physics:

a) [5 points] In the laboratory reference frame, what is the average \( \tau_{\pi_0} \) meson decay time? What is the average distance traveled by the \( \pi_0 \) meson in the laboratory reference frame before it decays?

The kinetic energy of the \( \pi_0 \) is \( K_{\pi_0} = 1.26\text{GeV} = (\gamma - 1) m_{\pi_0} \) where \( \gamma \) is the Lorentz factor. \( \gamma = \frac{1}{\sqrt{1 - \frac{1}{\beta^2}}} \). In the laboratory reference frame, the decay time constant is affected by Lorentz dilatation. \( \tau_{\text{Lab}} = \gamma \tau_{\pi_0} = 10^{-15} \text{s} \). The distance \( d \) traveled on average by the \( \pi_0 \) before it decays is \( d = \beta \cdot c \cdot \tau_{\text{Lab}} \) where \( \beta \) is the particle speed in unit of light speed \( c \). The Lorentz factor is \( \gamma = \frac{1}{\sqrt{1 - \frac{1}{\beta^2}}} \). This gives \( d \approx 3 \times 10^{-7} \text{m} \).

b) [6 points] From first principles, show that, in the reference frame where the \( \pi_0 \) is at rest, the two \( \gamma \) ray photons have the same energy equal to \( \frac{1}{2} m_{\pi_0} \cdot c^2 \).

1) Momentum must be conserved and, before the decay, in the rest frame, there is no momentum. So the two \( \gamma \) rays must have momenta that are equal and opposite. 2) \( \gamma \) ray photons are massless and their energy is equal to their momentum multiplier by \( c \). This tells us the two \( \gamma \) ray photons have the same energy. Finally, 3) energy must be conserved. Before the decay the energy is \( m_{\pi_0} \cdot c^2 \) and each gamma ray photon must have an energy of \( \frac{1}{2} m_{\pi_0} \cdot c^2 \).

c) [7 points] What is the energy range within which the photons should be detected in the laboratory reference frame?

We just have obtained the energy of the \( \gamma \) ray in the rest frame is \( E = \frac{1}{2} m_{\pi_0} \cdot c^2 \). With \( \theta \) the angle measured in the rest frame between the direction of the \( \gamma \) ray and that of the \( \pi_0 \), the component momentum of the \( \gamma \) ray along the direction of motion of the \( \pi_0 \) is \( p_x \cdot c = E \cos(\theta) \). In order to obtain the \( \gamma \) ray energy \( E' \) in the laboratory frame, we use a Lorentz transform:

\[
E' = (1 - \beta^2)^{-\frac{1}{2}} E + \beta E \sin(\theta) \]

It is clear that \( E' \) will be in the range from \( \frac{1}{2} \gamma m_{\pi_0} \sqrt{1 - \beta^2} \) to \( \frac{1}{2} \gamma m_{\pi_0} \sqrt{1 + \beta^2} \). With \( \gamma = 10 \) and \( \beta = 0.995 \), we get \( 3.5 \text{MeV} \leq E \leq 1.4 \text{GeV} \).

d) [7 points] Consider the \( \gamma \) ray photons that are emitted in the reference frame where the \( \pi_0 \) is at rest along a direction making an angle of \( \frac{\pi}{2} \) with the direction of motion of the \( \pi_0 \) meson. In the laboratory frame, what is the angle between the direction of these photons and that of the \( \pi_0 \)?

The transverse component of the momentum is unaffected by the Lorentz transform and

\[
P_y' \cdot c = P_y \cdot c = E \sin(\theta) \]

The momentum of the \( \gamma \) measured in the laboratory was calculated.
previously $P' \cdot c = E' = \gamma E (1 + \beta \cos(\theta))$. The angle measured between the direction of the

$\pi_0$ and $\gamma$ ray in the laboratory is given by $\sin(\theta') = \frac{P'_\gamma}{P'} = \frac{\sin(\theta)}{\gamma (1 + \beta \cos(\theta))}$. For $\theta = \frac{\pi}{2}$, this gives $\sin(\theta') = \frac{1}{\gamma}$. With $\gamma = 10$, we obtain $\theta' = \sin^{-1}(0.1) \approx 5.74^\circ$

as the angle for the $\gamma$ rays emitted at $\frac{\pi}{2}$ in the rest frame.
F) Two bodies → One body collision

* Before Collision ①  /  After Collision ②

\[ \begin{align*}
& \mathbf{m} \rightarrow \mathbf{m} \\
& \mathbf{M} \rightarrow \mathbf{M} \\
\end{align*} \]

* Classically

\[ \begin{align*}
E_1 &= \frac{1}{2} m \mathbf{v}^2 \\
E_2 &= \frac{1}{2} M \mathbf{v}^{'2} = M \mathbf{v}^{'2} \quad \text{since } M = 2m \\
P_1 &= M \mathbf{v} \\
P_2 &= M \mathbf{v}^{'} = 2m \mathbf{v}^{'}
\end{align*} \]

Conservation of momentum \( \Rightarrow M \mathbf{v} = 2m \mathbf{v}^{'}, \) \( \mathbf{v}^{'2} = \frac{\mathbf{v}^2}{2} \)

\[ \begin{align*}
E_2 &= \frac{m}{4} \mathbf{v}^2 \neq E_1 \quad \text{So the collision is non-elastic} \\
& \text{& some energy has been converted into heat, rotation, sound, deformation, ...}
\end{align*} \]

* Relativistically

\[ \begin{align*}
E_1 &= (\gamma m) c^2 \\
E_2 &= \gamma' M c^2 \\
P_1 &= \gamma M \mathbf{v} \\
P_2 &= \gamma' M \mathbf{v}^{'}
\end{align*} \]

Applying conservation of momentum \( \mathbf{v}^{'2} = \frac{\gamma M \mathbf{v}}{\gamma' M} \) & applying conservation of energy, we get \( \gamma' M = (1+\gamma)m \)

\[ \begin{align*}
\gamma' &= \frac{\gamma \mathbf{v}}{1+\gamma} & \text{& since } \gamma > 1, \quad \mathbf{v}^{'2} > \frac{1}{2} \\
\text{We also know that } & \quad M^2 c^2 = E^2 - P^2 c^2 \\
& \quad = (1+\gamma)^2 m^2 c^4 - \gamma M^2 \mathbf{v}^{'2} c^2 \\
& \quad M^2 = m^2 (1 + 2\gamma + \gamma^2 - \gamma^2 \frac{\mathbf{v}^{'2}}{c^2}) = m^2 (1 + 2\gamma + \gamma^2 \gamma^2) \\
& \quad \text{So } M = m \sqrt{2(\gamma+1)} > 2m \text{ since } \gamma > 1.
\end{align*} \]
This is more or less what happens in a nuclear fusion:

\[ ^2H + ^3H \rightarrow ^4He + n \]

\[ (M_{^2H} + M_{^3H} - M_{^4He} - M_n) c^2 = 17.6 \text{ MeV} \]

An other approach using Lorentz transform:

Let's write the four-momentum before collision.

\[
\begin{pmatrix}
E/c \\
\mathbf{P}_x
\end{pmatrix} = \begin{pmatrix}
(\gamma + 1)m_c \\
\gamma m_v
\end{pmatrix}
\]

Let's apply a Lorentz transform to get to the reference frame in which the spatial component, the momentum, be zero. Since, after collision, there is only one particle, this reference frame is the one in which this particle is at rest.

\[
\begin{pmatrix}
\gamma_0 - \beta \gamma_0 \\
-\beta \gamma_0
\end{pmatrix} \begin{pmatrix}
(\gamma + 1)m_c \\
\gamma m_v
\end{pmatrix} = \begin{pmatrix}
\gamma_0 (\gamma + 1)m_c - \beta \gamma_0 \gamma m_v \\
-\beta \gamma_0 (\gamma + 1)m_c + \gamma_0 \gamma m_v
\end{pmatrix} = \begin{pmatrix}
m_c \\
0
\end{pmatrix}
\]

The space (momentum) part gives us \( \beta = \frac{\gamma}{\gamma + 1} \frac{V}{C} = \frac{V}{C} \)

Knowing \( \beta \), we can calculate the corresponding \( \gamma_0 \):

\[
\gamma_0 = \left(1 + \beta^2\right)^{-1/2} = \left(1 - \frac{\gamma^2}{(\gamma + 1)^2} \beta^2\right)^{-1/2} = \left[\frac{(\gamma + 1)^2 - \gamma^2 \beta^2}{(\gamma + 1)^2}\right]^{-1/2} = \left[\frac{1 + 2\gamma + \gamma^2 (1 - \beta^2)}{(\gamma + 1)^2}\right]^{-1/2}
\]

\[
\gamma_0 = \left(\frac{\gamma + 1}{2}\right)^{1/2}
\]
We can now express $M$:

$$M = \left(\frac{\gamma + 1}{2}\right)^{1/2} \gamma \left(\frac{\gamma + 1}{2}\right)^{1/2} \gamma mc^2$$

$$M = mc \left(\frac{\gamma + 1}{2}\right)^{1/2} \left[ \gamma + 1 - \frac{\gamma^2 \beta^2}{\gamma + 1} \right]$$

$$M = mc \left(\frac{\gamma + 1}{2}\right)^{1/2} \left[ \frac{\gamma^2 (1-\beta^2) + 2\gamma + 1}{\gamma + 1} \right]$$

$$M = 2mc \left(\frac{\gamma + 1}{2}\right)^{1/2} = MC \sqrt{2(\gamma + 1)}$$

So we found for $MC$ & $\gamma$, the same expressions we had obtained earlier.

* Inversely, we could be thinking of the "explosion" of a particle at rest. 0 0 0 $M$ $\gamma$ 0 0 $\gamma$ $m$ $m$ which is similar to how we obtain energy from fission of heavy nuclei.

6) Compton scattering a 2 particle $\rightarrow$ 2 particles collision.

Before the scattering:

$$P_x = \left(\frac{E}{c} \cos \theta, \frac{E}{c} \cos \theta, 0, 0, 0, 0\right)$$

$$P_e = \left(Mc, 0, 0, 0, 0\right)$$

After scattering:

$$P_x' = \left(\frac{E}{c} \cos \theta, \frac{E}{c} \cos \theta, \frac{E}{c} \sin \theta, 0, 0, 0\right)$$

$$P_e' = \left(\gamma Mc, \gamma Mc \cos \theta, \gamma Mc \sin \theta, 0\right)$$
Conservation of energy & momentum write \( P_x + P_e = P'_x + P'_e \).

& this can be solved easily by doing \((P_x - P'_x)^2 = (P_e - P'_e)^2\)

\[ P_x^2 + P'_x^2 - 2P_x P'_x = P_e^2 + P'_e^2 - 2P_e P'_e. \]

\[
\begin{cases}
  P_x^2 = P'_x^2 = 0 & \& P_e^2 = P'_e^2 = m^2 c^2 \\
  P_x P'_x = \frac{E E'}{c^2} - \frac{E E'}{c^2} \cos \theta = \frac{E E'}{c^2} (1 - \cos \theta) \\
  P_e P'_e = \gamma m^2 c^2
\end{cases}
\]

So, using all this \( \frac{2 E E'}{c^2} (1 - \cos \theta) = 2(1 - \gamma)m^2 c^2 \).

Energy conservation gives \((1 - \gamma)m c = \frac{E}{c} - \frac{E'}{c}\).

& combining these two equations we get

\[
\frac{E E'}{c^2} (1 - \cos \theta) = m c \left( \frac{E}{c} - \frac{E'}{c} \right) = m (E - E')
\]

\[
\frac{1}{m c^2} (1 - \cos \theta) = \frac{E - E'}{E E'} = \frac{1}{E'} - \frac{1}{E}
\]

In quantum physics we will see that for photons \( E = \frac{h c}{\lambda} \) & \( E' = \frac{h c}{\lambda'} \) so \( \lambda' - \lambda = \frac{h c}{m c^2} (1 - \cos \theta) \).