III. Emergence of quantum mechanics.

A) Quantization of matter & change.
1) Quantization of matter.
   
   * 400 BC: Democritus, after Leucippus his teacher, developed a theory according to which the universe is made of void & atoms which combine in clusters forming the various substances we observe. Plato associated geometrical properties to atoms with just 5 elements (fire tetrahedron, air octahedron, water icosahedron, earth & South Cube).

   * 15th Century Spanish alchemist Geber develops corpuscularism. Each element consists of inner & outer layers of corpuscles. This was providing ground for searching for a transmutative production of gold.

   * 17th Century Boyle's law (inverse proportional relationship between volume & pressure for a gas at constant temperature) gives a first experimental basis for atomism & scientists like Robert Hooke, Pierre Gassendi & Rene Descartes attempted to account for various properties of matter in terms of atoms.

   * 1800 Fraunhofer studies dark lines in the solar light spectrum.

   * 1811 Amedeo Avogadro makes his famous hypothesis: "Equal volumes of ideal gas at the same temperature contain the same number of particles."
Avogadro's number, the constant \( \text{A} = 6.022 \times 10^{23} \text{ mol}^{-1} \) (1 mol = number of atoms in 12 g of Carbon). The name was associated to the constant after a suggestion in 1908 by Jean Perrin who was awarded the Nobel prize in 1926 in part for his work at determining this constant.

Avogadros's hypothesis was largely ignored, but it proved efficient at explaining changes in volume during chemical reactions between gases. The atomic nature of matter was soon established.

*1885* Johann Balmer (Switzerland) mathematician analyzed measurements of the hydrogen light spectrum made by Anders Angstrom and noticed the wavelength of the lines were 

\[ \frac{1}{\lambda} = \frac{B n^2}{n^2 - m^2} \]

In fact Balmer only observed lines for \( m = 2 \) & found \( B = 364 \text{ nm} \). Other series of lines were later observed:

- \( m = 2 \) Lyman Series UV 1906 - US
- \( m = 3 \) Paschen Series IR 1908 - German/Austrian
- \( m = 4 \) Brackett IR 1922 - US
- \( m = 5 \) Pfund IR 1924 - US

*1888* Johannes Rydberg (Swedish) writes the so-called Rydberg formula \( \frac{1}{\lambda} = R \left( \frac{1}{n^2} - \frac{1}{m^2} \right) \) for the spectral lines of alkali metals which was later explained by Bohr's model of the atom (1913) which we will study in detail.
2) Quantization of the Electric Charge

1834: Michael Faraday (English) studied electrolysis & announced his law on electrolysis:

1st: The mass of substance altered during electrolysis is proportional to the quantity of electricity (in Coulombs).

2nd: For a given quantity of electricity, the mass of substance altered during the electrolysis is proportional to the equivalent mass (now called molar mass).

Since matter is quantized, the suggested electricity is also quantized. Toward the end of the century, accumulating many experiments & using the kinetic theory of gases, it was found that the smallest quantum of charge involved in electrolysis was about \(10^{-20}\) C. (~1870)

1896: Pieter Zeeman was the assistant of Hendrik Lorentz at the University of Leiden (Netherlands). He built laboratory equipment to measure the splitting of spectral lines under the influence of a magnetic field. He got in trouble for not following his supervisor's instruction but was awarded the Nobel prize in 1902 for his observations and analysis, which established light was emitted by "oscillating" negatively charged particle with a mass estimated at a thousandth of that of the hydrogen atom.
*1897* Sir Joseph John (JJ) Thomson (British) awarded with the Nobel prize in 1906 for the discovery of the electron. He was working on the conductance of electricity in gases. He did a sequence of experiments establishing that "cathode rays" were deflected by electric & magnetic fields. Combining both aspects, he was able to measure the mass to charge ratio with a set-up similar to what is described here:

1. In a first phase $E$ & $B$ are adjusted so there is no deflection $qV = qE$ so $v = \frac{E}{B}$.

2. Then $B$ is turned off & one measures the deflection $y$.

   $y = \frac{1}{2} a t^2 + \frac{V_y}{B} t^2$ with $a = \frac{eE}{m}$ & since the deflection is small $t_1 \approx \frac{d}{v}$ & $t_2 \approx \frac{D}{v}$ & $V_y = at$,

   $y = \frac{1}{2} \frac{eE}{m} \frac{d^2}{v^2} + \frac{eE}{m} \frac{d D}{v^2} = \frac{eE d}{m v^2} \left( \frac{d}{2} + D \right) = \frac{e}{m} \frac{B^2}{E} d \left( \frac{d}{2} + D \right)$

For all the settings, the same value of $\frac{e}{m} = 1.710^{11}$ C/kg was obtained establishing the existence of a corpuscle with this property. I was later called the electron.
The Thomson experiment was then modified and became the mass spectrometer, which allows for very precise analysis of atomic species in even very small samples. 

\[ qVB = m \frac{v^2}{R} \] \[ \Rightarrow V^2 = \frac{qB^2R^2}{m} \]

Also \( \frac{1}{2}mv^2 = qV \) \[ \Rightarrow V^2 = \frac{2qV}{m} \]

and combining the two expressions:

\[ \frac{q}{m} = \frac{2V}{B^2R^2} \]

*1909 Robert Millikan (former student of A. Michelson) measured the electronic charge at the University of Chicago.

Method: One oil drop is selected by adjusting the electric field between the plates.

1. The electric field is turned off so the drop falls & quickly reaches a terminal velocity \( v \).

\[ F_{\text{drag}} = 6\pi \eta r^2 v = F_W = \frac{4}{3} \pi r^3 g (\text{Poit - Fair}) \]

\( \eta \) is the viscosity of the air, \( r \) the radius of the drop & \( \text{Poit and Fair} \) the densities of oil & air.

From this \( \frac{9}{2} \eta V^2 = 2g \frac{(\text{Poit - Fair})}{r} \) so \( r \) can be obtained by measuring the terminal velocity \( v \) & then the weight & buoyancy force \( F_W \) can be calculated.

2. The electric field is turned back on \( F_e = qE = q \frac{V}{d} = F_W \) at equilibrium.
In practice, equilibrium is difficult to achieve so instead a new terminal velocity (with the electric field on) is measured $v_E$ so $F_E - F_W = 6\pi \eta r v_E = \frac{F_W v_E}{v}$, so $g = \frac{d}{\sqrt{v}} F_W \left(1 + \frac{v_E}{v}\right)$.

This was repeated with many droplets and it could be shown the change came in integer multiples of an elementary charge $e_m = 1.55 \times 10^{-19}$ C which is less than 1% from the new well-measured charge of the electron $e = 1.602 176 487(40) \times 10^{-19}$ C. The error was due to Millikan using an incorrect value for the viscosity of the air.

**Note:** Richard Feynman in "Surely you are joking Mr. Feynman"Comment on how subsequent measurements of e indicated physicist can be influenced by previous results compared to their own.
B) Black body radiation

1) Introduction:
- Thermal radiation is emitted by all objects in relation to their temperature. Increased temperature corresponds to an increased accelerated motion of charged particles within atoms which therefore produce an electromagnetic radiation.

- A black body is an idealized object which absorbs all the electromagnetic radiation falling on it. The thermal radiation of a black body in thermodynamic equilibrium is known as the black body radiation.

- Radiation occurs over a broad range of frequency.

- In 1879, Joseph Stefan (Austrian) empirically established what is now known as Stefan-Boltzmann law according to which the power of the black body radiation per unit area (of the black body) is proportional to the fourth power of the temperature. The proportionality constant being a universal constant: \( \frac{dP}{dA} = \sigma T^4 \)
with \( \sigma = 5.6703 \times 10^{-8} \text{ W m}^{-2} \text{K}^{-4} \) Stefan's constant.

**Example 1** The Sun's emission quite closely follow the black body spectrum. On Earth we receive 1.3 kW m\(^{-2}\) from the Sun at a Distance \( D = 150 \times 10^9 \text{ m} \) with a radius \( R_0 = 7 \times 10^8 \text{ m} \). What is the temperature of the Sun?

The total power \( L = 4\pi D^2 \frac{dP}{dA} = 4\pi R_0^2 \sigma T_0^4 \), so

\[
T = \left( \frac{\frac{dP}{dA} D^2}{\pi R_0^2 \sigma} \right)^{1/4} = 5700 \text{ K}.
\]

**Example 2** Considering the Earth as a black body in thermal equilibrium, estimate the global temperature of our planet in terms of that of the Sun.

Thermal equilibrium implies as much power is radiated by Earth into space as is received from the Sun. Let's write this equation:

\[
\frac{4\pi R_0^2}{4\pi D^2} T_0^4 = \frac{4\pi R_0^2}{4\pi D^2} T_\oplus^4.
\]

So

\[
T_\oplus^4 = \frac{R_0^2}{D^2} T_0^4 \quad \text{and} \quad T_\oplus = \left( \frac{R_0}{D} \right)^{1/2} T_0^{1/2}.
\]

With \( R_0 = 7 \times 10^8 \text{ m} \), \( T_0 = 5700 \text{ K} \), and \( D = 150 \times 10^9 \text{ m} \), we get \( T_\oplus = 275 \text{ K} \approx 2 \text{ C} \).
In 1893 Wilhelm Wien (German) used thermodynamics applied to electromagnetic radiation in a cavity to derive the law named in his honor. It gives the wavelength of the maximum of the black body spectrum as being inversely proportional to temperature: \( \lambda_{\text{max}} = \frac{b}{T} \) with \( b = 2.898 \times 10^{-3} \text{ m.k.} \).

**Examples**

Sun \( T_\odot = 5700 \text{ K} \) \( \Rightarrow \lambda_{\text{max}_\odot} = 508 \times 10^{-9} \text{ m} = 508 \text{ nm} \) corresponding to green colour where most animals are the most sensitive vision.

Human body \( T_H = 37 \degree C + 273 \text{ K} = 310 \text{ K} \) so \( \lambda_H = 9.3 \text{ m} \) corresponding to the far infrared where IR cameras and also snakes are the most sensitive.

2) **Rayleigh - Jeans law**


- I am not sure how their demonstration exactly went but here is the idea:

- They thought an ideal black body could be thought of as a small hole giving into a radiation cavity.
They could count the number of electromagnetic oscillation modes for a given frequency & apply equipartition of energy associating kT to each mode (k is Boltzmann constant $k = 1.38 \times 10^{-23} \text{ J K}^{-1}$).

If the density of radiation energy at a given frequency is $U$ (inside the cavity), the power radiated through the hole per unit area is $\frac{dP}{dA} = \frac{1}{4} c U$ (at that same frequency that is).

So we could follow this program but let's first see the demonstration of $\frac{dP}{dA} = \frac{1}{4} c U$.

- Consider the small element of volume $dV = \pi r^2 dr \pi d\theta d\phi$.
- It contains an energy $U dV$.
- Some fraction of which will flow through the opening $dA$ during the following time $dt$ if $r < c dt$.
- The fraction of energy $U dV$ that may flow through $dA$ is $\frac{dA \pi r^2}{4\pi \pi^2}$.
- The power of radiation escaping through $dA$ is

$$dP = \frac{1}{dt} \int_0^{\pi/2} \int_0^{\pi/2} \int_0^{c dt} U \frac{dA \pi r^2}{4\pi \pi^2}$$
\[
\begin{align*}
\frac{dP}{dt} & \int \frac{C_d}{2} \frac{\pi}{2 \pi} \frac{1}{2} \frac{dA}{\pi} = \frac{CU}{4} \\
\frac{dP}{dt} & = \frac{CU}{4} \\
& \text{We now get back to Rayleigh-Jeans program}
\end{align*}
\]

let's start with a one dimensional stationary wave.
\[E = E_0 \sin \frac{2 \pi n x}{a} \sin \omega t = E_0 \sin k x \sin \omega t\]
with \(k = \frac{2 \pi}{\lambda}\) the wave number & \(\omega = 2 \pi f\) the angular frequency.

- In a one dimensional box of length \(a\)
  \[\lambda_1 = \frac{2a}{2} \quad \lambda_2 = \frac{2a}{3} \quad \lambda_3 = \frac{2a}{4} \quad \lambda_4 = \frac{2a}{5} \quad \ldots\]
  so \(k_n = \frac{2 \pi n}{2a} \quad \text{&} \quad \omega_n = \frac{2 \pi n f}{2a}\)

- In a 3-dimensional box
  \[E = E_0 \sin k_x x \sin k_y y \sin k_z z \sin \omega t\]
  satisfies the wave equation \(\nabla^2 E - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = 0\)
  if \(\omega^2 = c^2 \left( k_x^2 + k_y^2 + k_z^2 \right)\)
  For a cubic box of side \(a\)
  \[k_x = \frac{\pi n_x}{a} \quad k_y = \frac{\pi n_y}{a} \quad k_z = \frac{\pi n_z}{a}\]
  & each stationary mode composes
to a point labeled \((m_x, m_y, m_z)\) in the wave vector space.

- The number of stationary modes contributing to a frequency range from \(\omega\) to \(\omega + d\omega\) is simply the number of points between the spheres of radii \(\frac{\omega}{c}\) and \(\frac{\omega + d\omega}{c}\) with \(k_x > 0\), \(k_y > 0\), \(k_z > 0\).

- The density of modes is \((\frac{4\pi}{\lambda})^3\) & the number of modes between \(\omega\) & \(\omega + d\omega\) is \(dN = \frac{1}{4\pi} \frac{4\pi}{\lambda^2} \frac{d\omega}{c} \frac{a^3}{(\frac{\lambda}{c})^3} \frac{d\lambda}{\lambda^2}\). The extra factor 2 at the end comes from counting two polarizations for each mode. So \(dN = \frac{a^3}{c^3} \frac{d\omega}{\lambda^2} \). With \(f = \frac{\omega}{2\pi}\) & \(df = \frac{d\omega}{2\pi}\), \(dN = \frac{8\pi a^3}{c^3} \frac{f^2 df}{\lambda^4}\). & with \(\lambda = \frac{c}{f}\), \(d\lambda = \frac{c df}{f^2}\), we get \((f = \frac{c}{\lambda} \text{ & } df = \frac{c df}{\lambda^2})\), \(dN = \frac{8\pi a^3}{\lambda^4}\).

- Applying equipartition of energy with \(kT\) for each mode, the energy available within a wavelength interval \(d\lambda\) is \(\frac{dE}{d\lambda} = \frac{8\pi a^3}{\lambda^4} kT\), the energy density in the cavity is \(\eta = \frac{dE}{d\lambda a^3}\) & \(\frac{dP}{d\lambda} = \frac{2\pi c kT}{\lambda^4}\).
This clearly is not good:

- There is no maximum so it is not compatible with Wien's law.
- The integral diverges so it is not compatible with Stefan-Boltzmann's law.

This was referred to as the ultraviolet catastrophe.

3) Planck's Law

In 1900, a German physicist, Max Planck, realized that if he assumed the energy to come in "chunks" $E = nh\nu$ instead of being continuous, then the above calculation turns out to give the observed spectrum.

According to Boltzmann, the chance of having an energy $E_n$ in a given mode is $P_n = A e^{-E_n/kt}$ with Planck's hypothesis $P_n = A e^{-nh\nu/kt}$ where $A$ is a normalisation constant.

$$\sum_{n=0}^{\infty} P_n = A \sum_{n=0}^{\infty} e^{-nh\nu/kt} = 1$$

Remember that $P_0 = \sum_{i=0}^{p} q_i = \frac{1 - q}{1 - q} = 1 - q^{p+1}$.

So $$\frac{A}{1 - e^{-nh\nu/kt}} = 1 \quad \text{and} \quad A = 1 - e^{-nh\nu/kt}.$$ 

And $$P_n = (1 - e^{-nh\nu/kt}) e^{-nh\nu/kt}.$$
We can now calculate the average energy in a given mode

\[ \bar{E} = \sum_{n=0}^{\infty} E_n P_n = \sum_{n=0}^{\infty} n h \nu (1 - e^{-h \nu / kT}) e^{-n h \nu / kT} \]

\[ = h \nu (1 - e^{-h \nu / kT}) \sum_{n=0}^{\infty} n q^n \quad \text{with} \quad q = e^{-h \nu / kT} \]

We notice that

\[ \frac{dS_P}{dq} = \sum_{i=0}^{p} i q^{i-1} = \frac{-(p+1) q^p (1-q) + 1 - q^{p+1}}{(1-q)^2} \]

\[ \lim_{p \to \infty} \frac{dS_P}{dq} = \sum_{i=0}^{\infty} i q^{i-1} = \frac{1}{(1-q)^2} \quad \text{and} \quad \sum_{i=0}^{\infty} i q^i = \frac{q}{(1-q)^2} \]

So

\[ \bar{E} = h \nu (1 - e^{-h \nu / kT}) \frac{e^{-h \nu / kT}}{(1-e^{-h \nu / kT})^2} = \frac{\nu e^{-h \nu / kT}}{1 - e^{-h \nu / kT}} \]

\[ = \frac{hc e^{-hc/\lambda kT}}{\lambda (1 - e^{-hc/\lambda kT})} \]

We can now follow the same route as Rayleigh & Jeans:

The energy available within a wavelength interval d\lambda is

\[ \frac{dE}{d\lambda} = \frac{8 \pi a^3}{\lambda^4} \frac{hc e^{-hc/\lambda kT}}{\lambda (1 - e^{-hc/\lambda kT})} \]

& the corresponding energy density is \( U = \frac{dE}{d\lambda a^3} \)

The power per unit area radiated per wavelength interval in

\[ \frac{dP}{d\lambda} = \frac{2 \pi h c^2 e^{-hc/\lambda kT}}{\lambda^5 (1 - e^{-hc/\lambda kT})} \]

& finally

\[ \frac{dP}{d\lambda} = \frac{2 \pi h c^2}{\lambda^5 (e^{-hc/\lambda kT} - 1)} \]
\[ E = mc^2 \]

*In 1887, Heinrich Hertz (German) was producing electromagnetic waves using a spark gap. He noticed that when one of his spark gaps was open, the effect was even more violent when UV light was used.*

C) The photoelectric effect

*Which exactly determines the black body radiation can be derived. H in Planck's constant is obtained from entity Wien's law and Stefan's law.*
* When \( V < -V_0 \) the current is zero independently from the light intensity - \( V_0 \) is referred to as the stopping potential.

* \( V_0 \), the stopping potential depends on the light frequency linearly and there is a light frequency \( f_0 \) below which no photoelectric effect is observed.

! In 1905, Albert Einstein realized that Planck's hypothesis of quantisation of the electromagnetic radiation energy also provides a full explanation for the photoelectric effect.

* Consider \( \phi \) the work function of a metal - That is the energy required to pull out one electron.

<table>
<thead>
<tr>
<th>Element</th>
<th>Na</th>
<th>Ca</th>
<th>Cd</th>
<th>Al</th>
<th>Ag</th>
<th>Pt</th>
<th>Pb</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \phi \text{ (eV)} )</td>
<td>2.29</td>
<td>4.81</td>
<td>4.07</td>
<td>4.08</td>
<td>4.73</td>
<td>6.35</td>
<td>4.14</td>
</tr>
</tbody>
</table>

* If the energy delivered by electromagnetic radiation of frequency \( f \) comes in chunk \( hf \) then when \( hf < \phi \) no photoelectric effect is expected. \( f_0 = \frac{\phi}{h} \) is then the frequency threshold.

If \( hf > \phi \), \( hf - \phi \) is the kinetic energy of the extracted electron & the stopping potential in eV is \( V_0 = hf - \phi \).
\[ V_0 = \frac{\hbar^2}{4e} - \Phi e = \frac{\hbar}{e} \left( \frac{g}{h} - \Phi \right) = \frac{\hbar}{e} \left( \frac{g}{h} - \Phi_0 \right) \]

which corresponds well to the observed relationship between \( V_0 \) & \( g \).

- In 1923 Arthur Compton was able to observe the effect which was given his name, confirming at the same time the validity of both Planck's photon concept and special relativity.

**D) Rutherford's atomic model  a) Introduction**

* After his discovery of the electron, in 1904 started developing an atomic model in which electrons, small particles were arranged in rings surrounded by a diffuse positive charge - This was the plum pudding model.

* A few things were known:
  - Atoms contain electrons which are much lighter than atoms
  - They are neutral
  - They are small: 12 g of Carbon contains \( 6.02 \times 10^{23} \) atoms & diamond has a density of \( 3.5 \text{ g/cm}^3 \) so the volume occupied by one atom is \( \frac{12}{3.5 \times 6.02 \times 10^{23}} = 5.7 \times 10^{-24} \text{ cm}^3 \) suggesting an atomic size \( 1.7 \times 10^{-8} \text{ cm} \) which corresponds to 17 Å.

* Ernest Rutherford with Hans Geiger & Ernest Marsden at the university of Manchester were studying the radioactivity of Uranium and identified three types of particles they named \( \alpha, \beta \) & \( \gamma \).
They soon identified the α particles as twice ionized Helium by letting them accumulate in a glass tube through which they ran an electric discharge to record the characteristic light spectrum of the resulting fluorescence.

Rutherford then thought about using α particles as probes of the internal structure of atoms.

Most particles were deflected by a very small angle but a few were deflected by a large angle. Here is how this was described by Rutherford:

"It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you. On consideration, I realized that this scattering backward must be the result of a single collision, and when I made calculations, I saw that it was impossible to get anything of that order of magnitude unless you took a system in which the greater part of the mass of the atom was concentrated in a minute nucleus. It was then that I had the idea of an atom with a minute massive center carrying a charge."
b) Solid angles & cross sections

* A solid angle in the two dimensional "angular area" an object subtends from a given point. It measures how large the object appears to an observer at this given point.

* A being the area of a region of a sphere of radius r, the solid angle subtended by A at the center of the sphere is \( \Omega = \frac{A}{r^2} \). Solid angles are measured in steradians (sr).

  * If \( A \) covers the entire sphere then \( \Omega = \frac{4 \pi r^2}{r^2} = 4 \pi \) sr.
    (Note: There are \( 4.1252 \times 10^4 \) deg \(^2\) in \( 4 \pi \) sr & 1 deg \(^2\) = \( 3.04 \times 10^{-4} \) sr)

  * The solid angle \( d\Omega \) between \( \Theta \) & \( \Theta + d\Theta \) as on the figure is
    \[
    d\Omega = \frac{2 \pi r \sin \Theta d\Theta}{r^2} = 2 \pi \sin \Theta d\Theta
    \]

* Consider now a beam of particles all with the same energy & direction - The intensity of the beam is measured in \( m^{-2} \) sr\(^{-1}\).

* The beam is directed at a thin target of density \( p \) thickness \( x \) & area \( A \) - The number density of target atoms is \( n = \frac{p N_A}{M} \) where \( M \) is the molar mass & \( N_A = 6.02 \times 10^{23} \) is Avogadro's number. The number of atom targets within the beam is \( N = n A x = \frac{p N_A A x}{M} \).

* Beam particles are scattered by target atoms within a given element of solid angle \( d\Omega \) at a rate \( \frac{df}{d\Omega} \) - It is clear that \( \frac{df}{d\Omega} \) is proportional to both the beam intensity & the number of target in the beam \( N \).
The proportionality constant \( \frac{d\sigma}{d\Omega} \) is a characteristic of the interaction between beam particles and target scattering site,

\[
\frac{df}{d\Omega} = \frac{d\sigma}{d\Omega} \cdot I \cdot N = \frac{d\sigma}{d\Omega} \cdot I \cdot A \cdot \pi \cdot a^2.
\]

\( \frac{d\sigma}{d\Omega} \) is the differential cross section. From the definition, we see that \( \sigma \) is an area. In particle physics, a common unit of area for cross sections is the barn with \( 1 \text{ barn} = 10^{-28} \text{ m}^2 \).

c) Rutherford's scattering cross section

- Consider a central potential centered at 0 & a particle arriving along the x direction with an impact parameter b. The particle is scattered under an angle \( \theta \).

- The smaller the value of the impact parameter \( b \), the larger the scattering angle \( \theta \) & there must be a one to one relation between \( b \) and \( \theta \) so

the number of incoming particles between \( b \) & \( b + db \) must equal the number of particles scattered within \( d\Omega \) between \( \theta \) & \( \theta + d\theta \). So

\[ 2\pi I b \, db = \frac{d\sigma}{d\Omega} \cdot \frac{d\Omega}{d\sigma} \cdot 2\pi I \]

We have \( d\sigma = 2\pi I \sin \theta \, d\theta \), so

\[ 2\pi I b \, db = \frac{I}{d\sigma} \cdot 2\pi I \sin \theta \, d\theta. \]

\[
\frac{d\sigma}{d\Omega} = \frac{b}{\sin \theta} \cdot \frac{db}{d\theta} \quad \text{If we can establish} \quad \frac{db}{d\theta} \quad \text{for a given atomic model we will know} \quad \frac{d\sigma}{d\Omega} \quad \text{& obtain a prediction for the distribution of scattered particle to test the model.}
Let's calculate the relation between $b$ & $c$
assuming all the positive charge $Ze$ of the target atom is concentrated in a point at the center of the atom.

The magnitude of the Coulomb repulsion is 

$$F = \frac{3Ze^2}{4\pi \varepsilon_0 r^2}$$

The Coulomb force being radial, the angular momentum remains constant throughout the entire process. With $m$ the mass of the particle,

$$L = mrv \sin \varphi = m6v_0 = m r^2 \frac{d\varphi}{dt} \text{ so } r^2 = 6v_0 \frac{dt}{d\varphi}$$

The $y$ component of the electrostatic force gives us a differential equation for the $y$ component of the velocity,

$$F \sin \varphi = \frac{3Ze^2 \sin \varphi}{4\pi \varepsilon_0 r^2} = \frac{3Ze^2 \sin \varphi}{4\pi \varepsilon_0 bv_0} \frac{dv_y}{dt} = m \frac{dv_y}{dt}$$

Recognizing that $\frac{d}{dt} (\cos \varphi) = -\sin \varphi \frac{dv_y}{dt}$, we write

$$v_y (\varphi) = -\frac{3Ze^2}{4\pi \varepsilon_0 bv_0} \cos \varphi + C \text{ with } C \text{ an integration constant which we can set by realizing that } v_y (0) = 0$$

(when the projectile is infinitely far away before the collision it does not have any velocity along the $y$ direction.) So

$$v_y (\varphi) = \frac{3Ze^2}{4\pi \varepsilon_0 bv_0} (1 - \cos \varphi) \text{ for } \varphi \in [0, \pi - \theta]$$
Since energy is conserved the speed initial & final (when the projectile is at an infinite distance) are equal so we can write \( v_y (\pi - \theta) = v_0 \sin \theta \) or

\[
\frac{3Ze^2}{4\pi \varepsilon_0 b v_0 m} (1 - \cos (\pi - \theta)) = v_0 \sin \theta
\]

and, since \( \cos (\pi - \theta) = -\cos \theta \) we have

\[
\frac{3Ze^2}{4\pi \varepsilon_0 b v_0 m} = \frac{v_0 \sin \theta}{1 + \cos \theta} = v_0 \tan \left( \frac{\theta}{2} \right)
\]

which we can solve for \( b \) introducing \( E = \frac{1}{2} m v_0^2 \)

\[
b = \frac{3Ze^2}{8\pi \varepsilon E \tan \frac{\theta}{2}}
\]

\[
\frac{db}{d\theta} = \frac{3Ze^2}{16\pi \varepsilon E \sin^2 \frac{\theta}{2}} \quad \text{and} \quad \frac{d\sigma}{d\theta} = \frac{6}{\sin \theta} \cdot \frac{3Ze^2}{16\pi \varepsilon E \sin^2 \frac{\theta}{2}}
\]

Replacing \( b \) by its expression \( \frac{d\sigma}{d\theta} = \frac{\left[ \frac{3Ze^2}{16\pi \varepsilon E \sin^2 \frac{\theta}{2}} \right]^2}{2 \sin \theta \tan \frac{\theta}{2} \sin^2 \frac{\theta}{2}} \)

& using \( \sin \theta = 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \) we have in the end

\[
\frac{d\sigma}{d\theta} = \left[ \frac{3Ze^2}{16\pi \varepsilon E} \right] \frac{1}{\sin^4 \theta}
\]

With this expression of \( \frac{d\sigma}{d\theta} \), one can predict the number of particles that will be scattered in a given solid angle.

This is exactly how Rutherford established that all the positive charge and most of the mass of gold atoms are concentrated in a tiny region supposedly at the center of the atom.
The size of the nucleus

The nucleus does not have to be point like. For $\theta = 180^{\circ}$, the collision is head on. The $\alpha$ particle will get the closest to the nucleus

$$\frac{1}{2} M_{\alpha} v_{\alpha}^2 = \frac{9 x Q}{4 \pi \varepsilon_0 R_{\text{min}}}$$

so

$$R_{\text{min}} = \frac{1}{9} \frac{2 Q x Q}{M_{\alpha} v_{\alpha}^2}$$

with the 7.7 MeV $\alpha$ used by Geiger & Marsden on Gold.

$$R_{\text{min}} = 9 \times 10^{-9} \text{N m}^2 \text{C}^{-2} \cdot \frac{2 \times 7.4 \times (1.6 \times 10^{-19})^2}{7.7 \times 10^6 \times 1.6 \times 10^{-19}} = 3 \times 10^{-14} \text{m}$$

which was an upper limit on the radius of the gold nucleus.

Using less charged Aluminum, $Z = 13$, collisions were observed close to $\theta = 0$ and the estimated the nuclear radius to be $R = 10^{-14} \text{m}$. With $M_{\text{Al}} = 27 \times 1.7 \times 10^{-27} \text{kg}$, the density of nuclear material is

$$\rho = \frac{\frac{27 \times 1.7 \times 10^{-27} \text{kg}}{\frac{4}{3} \pi (10^{-14})^3}}{10 \text{kg m}^{-3}} = 10^{16} \text{kg m}^{-3}$$

This result was the beginning of nuclear physics.

E) Bohr's hydrogen atom

Once the idea of atoms consisting of a massive positive nucleus surrounded by a collection of electrons over a much larger scale, the question of the structure of the electron cloud came up.

One thing that comes to mind is a solar system like structure with circular orbits.

$$\frac{1}{4 \pi \varepsilon_0} \frac{Z e^2}{r^2} = \frac{M_{\text{Al}} v^2}{r}$$

One problem of this picture is that electrons radiate as they accelerate, so the orbits should not be stable.
In 1913, Niels Bohr (Danish physicist) made two assumptions:

- There are stationary orbits that do not radiate.
- Spectral lines associated with the emission and absorption of electromagnetic radiation result from transitions between orbits.

$$E_f = h\nu_f = |E_i - E_g|$$

$E_f > E_i$ corresponds to the absorption of $h\nu_f$

$E_i > E_f$ corresponds to the emission of $h\nu_f$.

Assuming a circular orbit, we can write the energy of the orbit of radius $r$:

$$E = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0} \frac{Ze^2}{r} = -\frac{1}{2} \frac{1}{4\pi\varepsilon_0} \frac{Ze^2}{r}.$$

Bohr then recognized that the Rydberg series corresponded to transitions between orbits whose angular momentum $L = mvr$ are integer multiples of $\frac{h}{2\pi}$.

So $mvr = n\frac{h}{2\pi}$ and $r = n\frac{h}{mv}$. 
\( r_n = \frac{n^2}{Z} \text{ a}_0 \) with \( a_0 = \frac{4\pi \varepsilon_0 \hbar}{m e^2} = 0.529 \text{ Å} \).

As is quite comparable to the expected radius of an atom given the mass density and molar mass of solid materials.

**Implication for the energy:**

\[
\begin{align*}
E_n &= -\frac{1}{2} \frac{1}{4\pi \varepsilon_0} \frac{Z e^2}{n^2} \left( \frac{m e^2}{4\pi \varepsilon_0 \hbar^2} \right) \\
&= -E_0 \frac{Z^2}{n^2} \quad \text{with} \ E_0 = \frac{1}{2} m \left( \frac{e^2}{4\pi \varepsilon_0 \hbar} \right)^2 = 2.18 \times 10^{-18} \text{ J} = 13.6 \text{ eV}
\end{align*}
\]

**Spectral lines.**

Bohr's hypothesis is that \( h\nu = E_{n_i} - E_{n_f} = -E_0 \frac{Z^2}{n_f^2} + E_0 \frac{Z^2}{n_i^2} \).

\( h\nu = \frac{hc}{\lambda} \quad \text{so} \quad \frac{1}{\lambda} = \frac{E_0 \frac{Z^2}{n_f^2}}{hc} \left( \frac{1}{n_f^2} - \frac{1}{n_i^2} \right) \)

which is exactly the form empirically obtained by Rydberg.

In the case of hydrogen \( Z = 1 \) & \( E_0 \frac{Z^2}{hc} = 1.098 \times 10^7 \text{ m}^{-1} \)

which is just Rydberg's constant.

The expression of \( \lambda \) for a given \( n_f \) & \( n_i \) reproduces also the spectrum of atoms of atomic number \( Z \) ionized \( Z-1 \) times so they have only one electron.

In 1914, James Franck & Gustav Hertz did an experiment to prove the energy levels of atoms with electrons. Their observation supported Bohr's model of the atom & they were awarded a Nobel prize in physics in 1925 while Bohr obtained the 1922 Nobel prize.