[1] Conduction electrons in a metal with a uniform static electric field

A uniform static electric field $E$ is established in a metal at uniform temperature with a conduction electron number density $n$ and relaxation time $\tau$. An electron is considered to undergo scattering at time $t = 0$ and then again at time $t$. Answer the following questions applying Drude’s theory of electrons in metals.

(a) What is the average energy lost by the electron in the second collision?

In a time $t$, the electrons acquire an energy $\dot{E} = \frac{eEt}{2m}$ on average. Since after the scattering at time $t$, the motion of the electron will be unrelated to the motion prior to the collision (as per Drude’s model hypothesis), all this energy must be lost to the ions of the crystal on average. So

$$\dot{E}_{\text{ion}}(t) = \frac{eEt}{2m}$$

(b) What is the average energy loss of the electron per collision?

The time $t$ of the second scattering is distributed with the probability density $\frac{dp}{dt} = \frac{1}{\tau} e^{-t/\tau}$ and the electron energy loss is $\dot{E}_{\text{ion}}(t) = \frac{eEt}{2m}$, so the average energy loss is

$$\langle \dot{E}_{\text{ion}} \rangle = \frac{eEt}{2m} \int_0^\infty \frac{1}{\tau} e^{-t/\tau} dt = \frac{(eEt)^2}{m} \quad \text{(Integration by parts. This was done in homework 7.)}$$

$$\langle \dot{E}_{\text{ion}} \rangle = \frac{(eEt)^2}{m}$$

(c) Consider a sample of cross-section area $A$ and length $L$ along which a current $I$ is flowing. Using your above result, demonstrate the total power dissipated is $P = RI^2$ and provide an expression for $R$.

The power dissipated in the sample is the mean energy loss per scattering of an electron multiplied by the number of electrons $n$ and divided by the mean time between successive scatterings for an electron.

So

$$P = \frac{A}{L} \frac{n}{\tau} \left( \frac{eEt}{m} \right)^2 = A n \sigma_0 E^2$$

with $\sigma_0 = \frac{n e^2 \tau}{m}$ the Drude model conductivity.

The current flowing through the sample is $I = A \sigma_0 E$, so $E = \frac{I}{A \sigma_0}$ and

$$P = \frac{L}{A \sigma_0} I^2 = RI^2$$

with

$$R = \frac{L}{A \sigma_0} = \frac{L m}{A n e^2 \tau}$$
(d) Suppose now that the metal has a uniform temperature gradient $\nabla T$. The average energy of an electron at temperature $T$ is $\epsilon(T)$. Show that the temperature gradient is responsible for an extra term proportional to $\nabla T \cdot E$ in the average energy loss of an electron per collision and provide the expression of the proportionality constant.

\[ \text{let } \bar{\epsilon}(t) \text{ represent the mean energy of an electron in a bath at temperature } T. \]

\[ \text{The energy loss per collision is } \bar{\epsilon}_{\text{loss}} = \bar{\epsilon}(t) - \bar{\epsilon}(0) \text{ where } \bar{\epsilon}(t) \text{ is the mean energy of the electron at time } t. \]

\[ \text{Suppose the first scattering at } t=0 \text{ occurs in } \mathcal{R}, \text{ then, on average, the second scattering at } t \]

\[ \text{occurs in } \mathcal{R} = \frac{t}{\text{m}} \frac{\epsilon E}{\text{m}} \text{ so we may write: } \bar{\epsilon}_{\text{loss}} = E \left( T(\mathcal{R} - \frac{t}{\text{m}} \frac{\epsilon E}{\text{m}}) \right) + \left( \frac{\epsilon E}{\text{m}} \right)^2 - E \left( T(\mathcal{R}) \right) \]

\[ \bar{\epsilon}_{\text{loss}} \propto E \left( T(\mathcal{R}) \right) \cdot \frac{t}{\text{m}} \frac{\epsilon E}{\text{m}} \frac{\partial E}{\partial T} + \left( \frac{\epsilon E}{\text{m}} \right)^2 - E \left( T(\mathcal{R}) \right) \]

\[ \bar{\epsilon}_{\text{loss}} \propto \frac{t}{\text{m}} \frac{\epsilon E}{\text{m}} \frac{\partial E}{\partial T} + \left( \frac{\epsilon E}{\text{m}} \right)^2 \]

(e) Consider again a sample of cross-section area $A$ and length $L$ with an electric potential difference $\Delta V$ and a temperature difference $\Delta T$ maintained between the two ends. Assuming the electric potential and temperature gradients are parallel and uniform, find a relation between $\Delta V$ and $\Delta T$ so the average energy loss per scattering cancels.

\[ \text{In these conditions } |\mathcal{E}| = \frac{\Delta V}{L} \text{ and } |\nabla T| = \frac{\Delta T}{L} \text{ and the energy loss cancelation gives:} \]

\[ \frac{\epsilon E}{\text{m}} \Delta T \frac{\partial E}{\partial T} = \left( \frac{\epsilon E}{\text{m}} \right)^2 \text{ or } \Delta V = \frac{1}{\epsilon} \frac{\partial E}{\partial T} \Delta T \]

\[ \text{In the stationary regime, when } \bar{\epsilon}_{\text{loss}} = 0 \text{ an electric potential appears between the two ends of a metallic rod maintained at a temperature difference } \Delta T. \text{ We have lost track of the sign but the higher electric potential is in the region of higher temperature since the negative electrons tend to move away from there.} \]

\[ \text{Applying the ideal gas model to the electron gas } \frac{\partial E}{\partial T} = \frac{3}{2} k_B \text{ and the predicted Seebeck effect is much larger than observed. This is mainly due to the fact electrons are fermions subject to the Pauli exclusion principle resulting in Fermi-Dirac statistics for which } \frac{\partial E}{\partial T} \text{ is smaller. Still, even if it is only qualitative, the Drude model of electrons in metals allows to obtain a physical picture of thermo-electric effects.} \]