Electron gas in two dimensions

We consider a two dimensional gas of electrons of mass \( m \) with \( N \) electrons confined to an area \( A \) so there is a density \( n = N/A \) of electrons per unit area.

(a) Express the Fermi wave vector magnitude \( k_F \) and the Fermi energy \( E_F \) in terms of \( n \)

\[ k_F = \sqrt{\frac{3\pi n}{\hbar^2}} \quad \text{and} \quad E_F = \frac{k_F^2}{2m} = \frac{\pi^2 k_F^2 n}{m} \]

(b) Express the density of levels \( g(E) \)

One way to do this is to write the density of states for the system at \( T = 0 \) and make the expression take the form:

\[ E = \frac{A}{\pi} \int_0^{E_F} g(E) \, dE \]

We can start from:

\[ E = \frac{A}{\pi} \int_0^{E_F} \frac{1}{2m} \frac{\hbar^2 k^2}{2m} \, dk \]

and with \( k = \frac{\sqrt{2mE}}{\hbar} \), \( \frac{dk}{dE} = \frac{\sqrt{2mE}}{\hbar} \)

\[ E = \frac{A}{\pi} \int_0^{E_F} \frac{m}{\hbar^2} E \, dE \]

The above approach is ok here because we are considering a very simple form of \( E(k) \). A more robust approach is to go back to the definition of \( g(E) \): It is the density of states per unit volume (area here) per interval of energy:

\[ g(E) = \frac{A}{\pi} \int_0^\infty \frac{k}{2m} \, dk \, \delta\left(E - \frac{k^2}{2m}\right) = \frac{4}{\pi} \int_0^\infty \frac{k}{2m} \, \delta\left(k - \sqrt{2mE/k}\right) = \frac{m}{\pi \hbar^2} \]

The properties of Dirac distribution:

\[ \int_0^\infty \delta(x) \frac{f(x-a)}{|f'(a)|} = 1 \]

Both approaches give the same result \( g(E) = \frac{m}{\pi \hbar^2} \) which is independent of the energy.

(c) Write the Sommerfeld expansion for \( n \) and conclude as to the relation between the chemical potential \( \mu \) and the Fermi energy \( E_F \)

At finite temperature, \( n = \int_0^{E_F} g(E) f(E, \mu) \) with \( f(E) = \frac{1}{e^{E/kT} + 1} \)

We can apply Sommerfeld's expansion:

\[ n \approx \int_0^{\mu} \frac{g(E) \, dE}{e^{E/kT}} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \left( \frac{kT}{\hbar} \right)^n \frac{\partial^n g(E)}{\partial E^n} \bigg|_{E=0} \]

but we just found that \( g(E) \) does not depend on \( E \), so \( \frac{\partial^n g(E)}{\partial E^n} = 0 \) and \( n = \frac{\mu}{kT} \frac{\partial g(E)}{\partial E} \bigg|_{E=0} \)

From \( T = 0 \) consideration, we also know that \( n = \int_0^{E_F} \frac{dE}{\pi \hbar^2} = \frac{\pi \hbar^2}{E_F} \)

These two result imply \( \mu = E_F \)
(d) Obtain a relation between $\mu$ and $E_F$ directly from the relation $n = \int_{-\infty}^{\infty} dE g(E) f(E)$ where $f(E)$ is the Fermi-Dirac occupation factor (Hint: proceed with the change of variable $x = e^{-(E-\mu)/k_B T}$). Write your result to the limit $k_B T \ll E_F$.

Proceeding as suggested, $n = \frac{m}{\pi h^2} \int_{0}^{\infty} dx \frac{1}{e^{x/k_B T} + 1}$ with the variable change $x = \frac{E-\mu}{k_B T}$, $dx = -\frac{1}{k_B T} dx$.

$$n = \frac{m}{\pi h^2} k_B T \int_0^{\infty} \frac{dx}{x} \frac{1}{e^{x/k_B T} + 1} = \frac{m}{\pi h^2} k_B T \int_0^{1/k_B T} \frac{dx}{x} \frac{1}{1 + e^{x/k_B T}}$$

Earlier we established $E_F = \frac{\pi^2 k_B T}{m} \Rightarrow E_F = k_B T \ln(1 + e^{\mu/k_B T})$.

Going to the low temperature limit, one might be tempted to do $1 + e^{\mu/k_B T}$ which leads to $E_F = \mu$ again so this is not good enough. Instead, we can write $E_F = k_B T \ln \left( e^{\mu/k_B T} (e^{\mu/k_B T} + 1) \right) = \mu + k_B T \ln \left( 1 + e^{\mu/k_B T} \right)$ and, to the limit $k_B T \ll \mu$, $E_F = \mu + k_B T e^{\mu/k_B T}$ so there is a difference but it is a very small one as $k_B T \ll \mu$ and $e^{\mu/k_B T} \ll 1$.

(e) Comment on the difference between your answers to question (c) and (d).

The Sommerfeld expansion is a Taylor expansion in powers of $k_B T$ around $k_B T = 0$. It simply fails to describe the term $k_B T e^{\mu/k_B T}$ which goes to zero in a non-analytic way as the successive derivatives of $f(x) = x e^{x/k_B T}$ all evaluate to 0 in $x = 0$. (you should try, it is kind of fun)