An alternate approach to problem #3, part d

We know from the previous problem that
\[ V_{xc} = \frac{4\pi e^2}{V} \left( \sum_{j=1}^{N_e} \frac{1}{|k - q_j|^2} \right) \]

So, as the summation runs over all modes within the cell, the sum is approximated as the following integral:
\[ V_{xc} = \frac{4\pi e^2}{V} \left( \sum_{j=1}^{N_e} \frac{1}{|k - q_j|^2} \right) = \frac{4\pi e^2}{V} \cdot \frac{V}{(2\pi)^3} \int \frac{d^3q}{|k - q|^2} \]

To work out the denominator, we apply the rules of the inner product:
\[ |k - q|^2 = \langle \vec{k}, \vec{q} \rangle - \langle \vec{k}, \vec{q} \rangle - \langle \vec{q}, \vec{k} \rangle + \langle \vec{q}, \vec{q} \rangle = k^2 + q^2 - 2kq \cos \theta \]

So, \[ V_{xc} = \frac{e^2}{2\pi^2} \int \frac{d^3q}{k^2 + q^2 - 2kq \cos \theta} \]

Now, recall the Legendre polynomials of spherical harmonics fame. Turns out, the generating function for these polynomials is given as:
\[ g(t, x) = (1 - 2tx + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \]

By factoring \( k^2 \) out of the integrand and letting \( x = \cos \theta \), we see that
\[ V_{xc} = \frac{e^2}{2\pi^2} \int \frac{d^3q}{k^2} \left[ \frac{1}{1 + 2(\frac{q}{k})x + (\frac{q}{k})^2} \right] = \frac{e^2}{2\pi^2} \int \frac{d^3q}{k^2} \left( \sum_{n=0}^{\infty} \left( \frac{q}{k} \right)^n P_n(x) \right) \]

Utilizing the linearity of the integral operator, we can express this as
\[ \frac{e^2}{2\pi^2} \sum_{n,m=0}^{\infty} \int \frac{d^3q}{k^2} \left( \frac{q^n}{k^n} P_n(x) \right) \left( \frac{q^m}{k^m} P_m(x) \right) = \frac{e^2}{2\pi^2} \sum_{n,m=0}^{\infty} \int \frac{d^3q}{k^2} \left( \frac{q^{n+m}}{k^{n+m}} \right) P_n(x) P_m(x) \]

We can then replace \( x \) with \( \cos \theta \) and write the integral in spherical coordinates
\[ V_{xc} = \frac{e^2}{2\pi^2} \sum_{n,m=0}^{\infty} \int \frac{d^3q}{k^2} \left( \frac{q^{n+m}}{k^{n+m}} \right) P_n(\cos \theta) P_m(\cos \theta) q^2 \sin \theta d\theta d\phi \]

Making the substitution \( u = \cos \theta \), \( du = -\sin \theta d\theta \) and separating the integral,
\[ V_{xc} = \frac{e^2}{2\pi^2} \sum_{n,m=0}^{\infty} \int \left( \frac{q^{n+m}2}{k^{n+m}} \right) du \int_0^{2\pi} d\phi \int_0^1 P_n(u) P_m(u) du \]

Making the substitution \( u = \cos \theta \), \( du = -\sin \theta d\theta \) and separating the integral,
Note that the reversal of the limits of integration on the last integral absorbs
the negative sign generated from the substitution.

Now, the Legendre polynomials are orthogonal and satisfy the relation

\[
\int_{-1}^{1} P_n(x) P_m(x) \, dx = \frac{2}{2n+1} \delta_{mn}
\]

Therefore, upon evaluating the integral we have

\[
V_{xc} = \frac{e^2}{2\pi} \sum_{n,m} \left( \frac{1}{n+m+3} \right) \left( \frac{k_f}{k_{n+m+2}} \right) \left( \frac{2}{2n+1} \right) \delta_{mn}
\]

Applying the Kronecker delta and pulling a \( k_f \) out of the numerator,

\[
V_{xc} = \frac{e^2 k_f}{\pi} \sum_{n} \left( \frac{1}{n+3} \right) \left( \frac{2}{2n+1} \right) \left( \frac{k_f}{k} \right)^{2n+2}
\]

Letting \( x = \frac{k_f}{k} \) and using either a table of series or software, the sum evaluates to

\[
V_{xc} = \frac{e^2 k_f}{\pi} \left( \frac{x - \tanh^{-1}(x) + x^2 \tanh^{-1}(x)}{x} \right) = \frac{e^2 k_f}{\pi} \left[ x^{-1} \tanh^{-1}(x) + 1 \right]
\]

Finally, using the fact that \( \tanh^{-1}(x) = \frac{1}{2} \ln \left( \frac{1+x}{1-x} \right) \) \( |x| > 1 \), we arrive at

\[
V_{xc}(k) = \frac{e^2 k_f}{\pi} \left[ \frac{x^{-1}}{2x} \ln \left( \frac{1+x}{1-x} \right) + 1 \right] = \frac{e^2 k_f}{\pi} V_{xc}(k_f)
\]