Scale relativistic formulation of non-differentiable mechanics I
Application to the harmonic oscillator

Mei-Hui Teh · Laurent Nottale · Stephan LeBohec

Received: date / Accepted: date

Abstract This article is the first in a series of two presenting the scale relativistic approach to non-differentiability in mechanics and its relation to quantum mechanics. In this first paper, we present the definition of a complex scale-covariant time-differential operator and show that mechanics of non-differentiable paths is implemented in the same way as classical mechanics but with the replacement of the time derivative and velocity with the time-differential operator and associated complex velocity. With this, the generalized form of Newton’s fundamental relation of dynamics is shown to take the form of a Langevin equation in the case of stationary motion characterized by a null average classical velocity. The numerical integration of the Langevin equation in the case of a harmonic oscillator reveals the same statistics as the stationary solutions of the Schrödinger equation for the same problem. This motivates the second paper which makes the relation to quantum mechanics explicit by discussing the axioms of quantum mechanics in the Scale Relativity framework.

Keywords Scale Relativity · Foundation of Quantum Mechanics · Stochastic Mechanics

1 Introduction

The relativity principle prescribes the laws of physics to have the same expression in all reference systems. It is implicitly understood that the relation between two reference systems is entirely specified by their relative position,
orientation, and motion. The Galilean implementation of the relativity principle with the least action principle establish the whole classical mechanics [?]. The special relativistic implementation with the identification of the speed of light as an invariant under the Lorentz transformations of the coordinates allows for a simple and natural theory of electrodynamics. Galilean and special relativity apply to the class of inertial reference systems, all in uniform relative motion with respect to each other. The transformation of the coordinates from one reference system to another is then linear. The next step consists in extending the relativity principle to non inertial reference systems. Coordinates transformations are then not necessarily linear anymore but they are at least twice differentiable. The least action principle takes the form of the geodesic principle and, with the equivalence principle, this leads to considering gravitation as a manifestation of the curvature of space-time[?].

From a purely geometrical point of view, the next step toward greater generality would be to consider transformations that are still continuous but not necessarily differentiable. The task of implementing such a generalization of the relativity principle is absolutely formidable. However, there are two foreseeable direct implications of this idea. First, any two points would be connected by an infinite number of geodesics. Consequently, determinism, implemented by the principle of least action, has to give way to a probabilistic and statistical description. Second, geodesics are still continuous but they are non-differentiable. This implies their scale dependent and divergent nature, that is fractal in a general meaning[?]. This identifies scaling laws as the essential tool for the implementation of a non-differentiable relativity. This proposal, known as Scale Relativity, was originally formulated by one of the authors [?].

The consideration of non-differentiability and the associated scale dependence may provide new insights in two major aspects of the physical world. First, in standard quantum mechanics, the scale dependence is explicit in the Heisenberg uncertainty relations and several authors [?,?]? commented on the fractal nature of the quantum path. Second, complex and chaotic systems often involve the coupling between phenomena occurring at different scales and always demonstrate structures over broad ranges of scales. There is no general framework for modeling these systems while chaos is a universal phenomenon investigated in many domains of science [?].

The project of abandoning the hypothesis of differentiability seems daunting as differential calculus is the main mathematical tool of physics. We may however proceed without abandoning differential analysis tools by smoothing out any non-differentiable structures smaller than some parametric scale. In this paper we demonstrate this approach in the case of non-differentiable space coordinates while maintaining time as an external and absolute parameter. Furthermore, we are restricting ourselves to the type of non-differentiable paths with the scale dependence corresponding to quantum mechanical path or Brownian motion as motivated by Section ???. Within this framework, in Section ??, we define a time-differential operator. In Section ??, applying the stationary action principle, we show that the transition from differentiable
to the considered non-differentiable paths is simply implemented by replacing the usual time derivative with this new time-differential operator while keeping track of changes in the Leibniz product rule, which result from a higher order differential term. This extends the principle of covariance to scaling laws with the new time-differential operator playing the role of a scale-covariant derivative. In Section ??, we show that, under the restriction to stationary solutions, the fundamental relation of dynamics generalized to non-differentiable paths takes the form of a Langevin equation. We then proceed to the numerical integration of the Langevin equation in the case of a simple harmonic oscillator and show we recover the statistic of quantum mechanics for the same system. Finally, Section ?? summarizes the results and provides a discussion and an outlook on their implications.

2 Fractal dimension

We may approach the concept of fractality from the point of view of physical measurements. The measurement of any quantity $Q$ amounts to counting the number $M_{\delta Q}$ of times the unit quantity $\delta Q$ fits in $Q$. The result of the measurement is then noted $Q = M_{\delta Q} \cdot \delta Q$. The measurement unit $\delta Q$ is generally chosen in a way that closely relates to the precision with which the measurement is carried out. For this reason and by simplification, we do not distinguish the resolution from the measuring unit.

In practice, it is usually implicitly assumed that two measurements of the same quantity performed with different resolutions $\delta Q$ and $\delta Q'$ are related by $M_{\delta Q} \cdot \delta Q \approx M_{\delta Q'} \cdot \delta Q'$ up to the experimental errors. The presumption then is that the quantity $Q$ does not depend on the scale of inspection. This leads to the idea that the measurement accuracy is improved by the use of a measuring device with an improved resolution.

This logic however breaks down when the structures contributing to the measurement outcome themselves depend on the scale of inspection. This is better seen when the quantity $Q$ is of geometrical nature, such as a length, an area or a volume. One may then write the resolution as $\delta Q = (\delta x)^D_T$ where $\delta x$ is the length scale with which the object is inspected (for simplification, we assume this scale to be the same in all directions, which is not necessarily the case) and $D_T$ is the topological dimension of the quantity being measured. Comparing measurements carried out with different resolutions, one may write $(M_{\delta x}/M_{\delta x'}) = (\delta x/\delta x')^{-D_F}$ with the fractal dimension $D_F = D_T + \delta D$, where $\delta D$, an anomalous exponent, is introduced to account for the resolution or scale dependance of the measurement. With this, $Q_{\delta x}/Q_{\delta x'} = (\delta x/\delta x')^{-D_F} (\delta x/\delta x')^{D_T} = (\delta x/\delta x')^{D_T-D_F} = (\delta x/\delta x')^{-\delta D}$. This can be inverted to establish the fractal dimention of an object as $D_F = D_T - \frac{\log Q_{\delta x}/Q_{\delta x'}}{\log \delta x/\delta x'}$.

We may apply this to the case of the measurement of the length, $D_T = 1$, of the von Koch curve (See Figure ??). Take $\delta x/\delta x' = 3^p$, with $p$ some integer. Then by the construction of the von Koch curve, $M_{\delta x}/M_{\delta x'} = 4^{-p}$.
Fig. 1 Left panel: Five iterations in the construction of the von Koch curve. Each iteration consists in replacing each segment of the previous iteration by 4 segments of length 3 times smaller, ultimately resulting in a curve of length diverging with resolution at a rate characterized by the fractal dimension $D_F = \frac{\ln 4}{\ln 3}$. Alternatively, the successive curves can be regarded as representations of a same object inspected with different resolutions. Right panel: Cavalier projection of five iterations in the construction of a curve in three dimensions. Each segment of the previous iteration is replaced by 9 segments of length 3 times smaller, resulting in a fractal dimension $D_F = \frac{\ln 9}{\ln 3} = 2$.

so that $Q_{ss}/Q_{ss'} = (3/4)^p$ and, $D_F = \log 4 / \log 3 \approx 1.261859 \cdots$. The fact that $D_F > D_T$ indicates the divergence of the length as the curve is inspected at smaller scales. The right panel of Figure ?? presents another example with a fractal dimension $D_F = 2$. In self-similar objects such as the curves in Figure ??, the fractal dimension remains constant under successive changes of scale. This, however, is not necessarily the case, and curves with a fractal dimension that depends on scale are easy to imagine. We can even imagine changes of scale resulting in transitions between fractal and non fractal regimes. A very good example of this is the Brownian motion of a particle. The inspection of the trajectory with a fine enough resolution reveals the kinks resulting from collisions with individual molecules of the surrounding fluid. Between collisions, the particle is observed in the ballistic regime, with a path of fractal
dimension 1.0, while at poorer resolution, in the diffusive regime, we will soon see that the fractal dimension of the path is 2.0.

It is useful to further concentrate on the specific case of the measurement of the length $\mathcal{L}$ of a curve. From what precedes, when a path is inspected with two different resolutions $\delta x$ and $\delta x'$, we have $\mathcal{L}_{\delta x} / \mathcal{L}_{\delta x'} = (\delta x / \delta x')^{1-D_F}$. Instead of measuring the length of the curve by inspection at a given resolution $\delta x$ or $\delta x'$, we may consider it being traveled at a constant rate and inspected at regular time intervals $\delta t$ or $\delta t'$. For this, we can consider a section of the path to be traveled in a given time $T = \delta t (\mathcal{L}_{\delta x} / \delta x) = \delta t' (\mathcal{L}_{\delta x'} / \delta x')$, in such a way $\delta t$ and $\delta t'$ are the times required to travel distances $\delta x$ and $\delta x'$ respectively. This implies $\delta t / \delta t' = (\delta x / \delta x') (\mathcal{L}_{\delta x'} / \mathcal{L}_{\delta x}) = (\delta x / \delta x')^{D_F}$ and $\mathcal{L}_{\delta t} / \mathcal{L}_{\delta t'} = (\delta t / \delta t')^{\frac{1}{2-D_F}}$.

With this, we can consider the case of Brownian motion observed at scales corresponding to the diffusion regime. The distance traveled during a given time interval $\delta t$ scales with $\sqrt{\delta t}$ and the total distance traveled over the full duration $T$ of the observation is proportional to $\sqrt{\delta t T / \delta t} = T (\delta t)^{-1/2}$, corresponding to $D_F = 2$, which implies a divergence of the velocity at small scale until the ballistic regime is reached.

We can also consider the case of the path of a quantum particle. In 1965, Feynman and Hibbs wrote "It appears that quantum-mechanical paths are very irregular" and "...the 'mean' square value of a velocity averaged over a short time interval is finite, but its value becomes larger as the interval becomes shorter" in such a way that "...although a mean square velocity can be defined, no mean square velocity exists at any point. In other words, the paths are non-differentiable"[2]. It should be noted that this preceded the word fractal coined by Benoît Mandelbrot in 1975. In order to see this quantitatively from a simple argument [2] based on the Heisenberg uncertainty relations, consider a particle of mass $m$ whose position is measured at regular time intervals $\Delta t$. If the particle is at rest on average, the measured path it travels results from the quantum fluctuations $\delta p$ of the momentum so that after $N$ measurements, the length of path that will have been recorded on average will be $\langle \mathcal{L} \rangle = N \frac{\hbar}{m} \Delta t$. Using the Heisenberg uncertainty relation $\delta p \cdot \delta x \approx \hbar/2$, this may be rewritten $\langle \mathcal{L} \rangle = N \frac{\hbar^2}{m^2} \Delta t$ and identifying with $\langle \mathcal{L} \rangle \propto (\delta x)^{1-D_F}$ we see that $D_F = 2$. From one measurement to the next, the smallest significant change in path length $\langle \Delta \mathcal{L} \rangle$ is the position measurement resolution itself $\langle \Delta \mathcal{L} \rangle \approx \delta x$. With this, we obtain $\delta x^2 \approx \frac{\hbar^2}{m^2} \Delta t$. Comparing this with $\delta t \propto \delta x^{D_F}$ we identify the effective diffusion coefficient $\mathcal{D} = \frac{\hbar}{2m}$.

In the following section, we establish a general method of approach to these non-differentiable and scale dependent paths and we apply this method specifically to the case $D_F = 2$ because of it’s importance in physics.

3 Scale covariant time-differential operator

The usual derivative $f'(t)$ of a function $f(t)$ is defined by considering the change of the value of the function in the limit of infinitesimally small increase
Fig. 2  The black curve represents a non-differentiable path $x(t)$. It was obtained by dichotomy iterations. The red line represents the first iteration, the blue line the second, the green line the third and the black line the ninth. When going from one iteration to the next, the position $x$ of the midpoint of each segment is shifted randomly in such a way that, on average, the combined displacements corresponding to the two resulting segments is $\sqrt{2}$ larger than the displacement represented by the original one. This corresponds to a fractal dimension of 2.0. As the curve is considered with finer resolution, the left and right derivatives in $t = 0$ for example keep fluctuating with an amplitude that diverges.

$\left( + \right)$ or decrease $\left( - \right)$ of the argument:

$$
\begin{align*}
 \lim_{\delta t \to 0^+} f'(t) &= \lim_{\delta t \to 0^+} \frac{f(t + \delta t) - f(t)}{\delta t} \\
 \text{and} \quad \lim_{\delta t \to 0^+} f''(t) &= \lim_{\delta t \to 0^+} \frac{f(t) - f(t - \delta t)}{\delta t}
\end{align*}
$$

with $f'(t) = f'_+(t) = f'_-(t)$ as long as the function is differentiable in $t$. If, on the contrary, the function is continuous but non-differentiable in $t$, then $f'(t)$ is no longer uniquely defined as $f'_+(t) \neq f'_-(t)$. Furthermore, if the function is non-differentiable in a dense set of points (a set is said dense if any neighborhood of any point not in the set contains at least one point in the set), then, taking the limit $\delta t \to 0^+$ becomes impossible in practice as the outcome fluctuates indefinitely with a diverging amplitude as the limit is approached (see Figure ??). We then have to refrain from taking the limit and define a
double-valued and explicitly scale dependent differential, which can be applied to a scale dependent function $f(t, \delta t)$:

$$f'_+(t, \delta t) = \frac{f(t + \delta t, \delta t) - f(t, \delta t)}{\delta t} \quad \delta t > 0;$$

$$f'_-(t, \delta t) = \frac{f(t + \delta t, \delta t) - f(t, \delta t)}{\delta t} \quad \delta t < 0.$$  

With this in mind, we can consider the displacement of a point of position $x(t)$ along a non-differentiable path with the following representation with two terms:

$$dx_+ = v_+ dt + db_+ dt > 0$$

$$dx_- = v_- dt + db_- dt < 0$$  

The first term proceeds from a *usual velocity* $v_\pm$, which depends on the scale of inspection (see Figure ??). By *usual velocity* we mean a finite displacement $\delta x$ divided by the time $\delta t$ it takes to complete it. The displacement $\delta x$ or the time $\delta t$ correspond to the scale or resolution with which the path is inspected to yield a specific representation. On the contrary, $b_\pm$ represents the residual, possibly a stochastic process, to be revealed by finer observations. There is an infinite number of paths whose inspection at a finite resolution $\delta t$ yield the same representation. They differ from each other only by their respective $db_\pm$. Consequently, the expectation value of $db_\pm$ over this ensemble must cancel $\langle b_\pm \rangle = 0$. Indeed, the non cancelation of this expectation value would betray some knowledge about the path inspected with a resolution finer than actually considered. We will shortly come back to the statistical properties of $b_\pm$ as they determine how the path representation is affected by a change in resolution. Indeed, the *usual velocity* at a given resolution $\delta t$ can be seen as deriving from the residual $b_\pm$ in the representation of the same path inspected at a coarser resolution. It is worth stressing that this representation of the path with two terms is a formalization of the usual way we deal with trajectories: the details smaller than some resolution scale are disregarded or smoothed out. This can be regarded as the promotion of resolution to be one of the relative characteristics of reference frames, at the same level as position, orientation and motion in a way which constitutes the essence of the Scale Relativity approach[?].

With this, it is convenient to define the *classical* time-differentials as the expectation values of the time-differentials after and before the considered point:

$$\frac{d_+}{dt} x = v_+ + \langle \frac{db_+}{dt} \rangle = v_+ \quad \text{and} \quad \frac{d_-}{dt} x = v_- + \langle \frac{db_-}{dt} \rangle = v_-$$

Furthermore, rather than continuing to manipulate the two classical differential operators separately, we combine them linearly into a single complex time-differential operator[?].
\[
\frac{\dot{d}}{dt} = \frac{1}{2} \left( \frac{d_+}{dt} + \frac{d_-}{dt} \right) - i \frac{1}{2} \left( \frac{d_+}{dt} - \frac{d_-}{dt} \right).
\]  

(2)

The real part corresponds to the average of the after and before differentials. It can be thought of as the classical differential, which is preserved for a differentiable path in the limit \( dt \to 0 \). The imaginary part is the halved difference between the after and before differentials. It can be thought of as the kink differential which vanishes for a differentiable path in the limit \( dt \to 0 \).

When acting on \( x \) with the complex time-differential operator, we can define the complex velocity:

\[
\mathbf{V} = \frac{\dot{d}}{dt} \mathbf{x} = \frac{\mathbf{v}_+ + \mathbf{v}_-}{2} - i \frac{\mathbf{v}_+ - \mathbf{v}_-}{2} = \mathbf{V} - i \mathbf{U}
\]

where \( \mathbf{V} \) can be regarded as the classical velocity and \( \mathbf{U} \) is an additional term, the kink velocity which persists under the inspection of non-differentiable paths with ever finer resolutions.

Equipped with these definitions, we can consider the time-differential of a regular field \( f(x(t), dt, t) \), along a possibly non differentiable path \( x(t, dt) \), described in terms of a finite resolution representation (Equation ??). The total derivative of \( f(x, t) \) is written as (the repetition of an index means the implicit summation over that index):

\[
\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} dx_i \frac{dt}{dt} + \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} \langle db_i \cdot db_j \rangle \frac{dt}{dt} + \frac{1}{6} \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k} \frac{dx_i dx_j dx_k}{dt} + \cdots
\]

At this point we need to make a choice as to the statistical nature of the residual process \( b_{\pm} \). As we would like to concentrate on the case \( D_F = 2 \), specific to Brownian motion and quantum mechanical paths, we write it as a Wiener process with \( \langle db_{\pm} \rangle = 0 \) as already discussed, \( \langle db_{i+} \cdot db_{i-} \rangle = 0 \), and \( \langle db_{i+} \cdot db_{j+} \rangle = \langle db_{i-} \cdot db_{j-} \rangle = 2D \delta_{i,j} dt \) with \( D \) akin to a diffusion coefficient. It should be stressed that this choice is very specific. It corresponds to describing Markovian or random walks. The underlying statistical process can naturally be thought as Gaussian by virtue of the central limit theorem. However any other statistical distribution would be equally valid and would make no difference for the rest of the development. We now consider both the after and the before time-differentials while keeping only the terms that do not vanish with \( dt \). Note that in taking the limit here, we do not change the scale as we are now considering a specific representation of the path with a classical component linear in \( dt \) and a stochastic component accounted for through an expectation value with a fractional power of \( dt \):

\[
\frac{d_+ f}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x_i} v_{i+} \frac{dt}{dt} + \frac{\partial f}{\partial x_i} \langle db_{i+} \rangle \frac{dt}{dt} + \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j} \langle db_{i+} db_{j+} \rangle \frac{dt}{dt}
\]
Since $b_\pm$ is of order $\sqrt{dt}$, the third term would diverge with $dt \to 0$ if it were not for the expectation value, which makes it cancel as $\langle db_\pm \rangle = 0$. The after and before time-differentials become:

$$d_\pm dt f = \frac{\partial f}{\partial t} + v_\pm \cdot \nabla f \pm D \nabla^2 f$$

Combining the after and before differential operators in a single complex differential operator as before:

$$\hat{dt} = \frac{\partial}{\partial t} + V \cdot \nabla - iD \Delta$$

It can be anticipated that going from differentiable geometry to non-differentiable geometry should be implemented by replacing the usual time derivative $\frac{d}{dt}$ with this complex time-differential operator $\hat{dt}$ provided proper attention is given to the changes in the Leibniz product rule implied by the second derivative appearing in the last term of Equation ??, In the next section, we verify that this is indeed the case for the Lagrange formulation of mechanics.

4 Mechanics of non-differentiable paths

Considering non-differentiable paths with the corresponding double-valued velocities with after and before components $x_\pm$, we assume here that the mechanical system with the configuration coordinate $x$ can be characterized by a now complex Lagrange function $\mathcal{L}(x, V, t)$. We can then express the action for the evolution of the system between times $t_1$ and $t_2$ as

$$S = \int_{t_1}^{t_2} \mathcal{L}(x, V, t) dt.$$ In order to lighten the notation without losing any generality, we proceed by considering a one-dimensional problem with $V = \frac{1}{2}(\dot{x}_+ + \dot{x}_-) - \frac{i}{2}(\dot{x}_+ - \dot{x}_-)$ and $\mathcal{L}(x, V, t) = \mathcal{L}(x, \frac{1-i}{2} \dot{x}_+ + \frac{1+i}{2} \dot{x}_-, t)$. The stationary action principle can be written as

$$\delta S = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}_+} \delta \dot{x}_+ + \frac{\partial \mathcal{L}}{\partial \dot{x}_-} \delta \dot{x}_- \right) dt = 0.$$ Provided the complex Lagrange function is an analytic function of $V$ and making use of $\frac{d}{dt}$ (See equation ??), it becomes:

$$\delta S = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial V} \frac{1-i}{2} \delta \dot{x}_+ + \frac{\partial \mathcal{L}}{\partial V} \frac{1+i}{2} \delta \dot{x}_- \right) dt = \int_{t_1}^{t_2} \left( \frac{\partial \mathcal{L}}{\partial x} \delta x + \frac{\partial \mathcal{L}}{\partial V} \frac{\hat{dt} \delta x}{dt} \right) dt = 0$$

We recover the usual form of the expression of action stationarity with the velocity replaced with our complex velocity $V$ and the time derivative replaced with our complex time-differential $\hat{dt}$. From this point, we need to integrate by parts. This requires care as $\frac{d}{dt}$ includes a second derivative which affects
the product rule. One can verify that \( \frac{d}{dt}(f \cdot g) = \frac{df}{dt}g + f\frac{dg}{dt} + 2iD \nabla f \cdot \nabla g \) and we obtain

\[
\delta S = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial x} \delta x + \frac{\dot{d}}{dt} \left( \frac{\partial L}{\partial V} \delta x \right) - \frac{\dot{d}}{dt} \left( \frac{\partial L}{\partial V} \right) \delta x \right) dt = 0,
\]

where the term in \( \frac{dx}{dt} \) was discarded as \( \delta x \) is not a function of \( x \). Considering that \( \delta x(t_1) = \delta x(t_2) = 0 \) and requiring this equation to hold for any infinitesimal \( \delta x(t) \), we obtain the usual Euler-Lagrange equation but with the complex time-differential operator and velocity:

\[
\frac{\partial L}{\partial x} - \frac{\dot{d}}{dt} \left( \frac{\partial L}{\partial V} \right) = 0
\]

Using the usual form of the kinetic energy and including a purely real potential energy term \( \Phi \) associated with a conservative force acting on the particle: \( L = \frac{1}{2}mV^2 - \Phi \), the Euler-Lagrange equation results in a generalized form of Newton’s relation of dynamics

\[
m\frac{\dot{d}}{dt}V = -\nabla \Phi. \tag{4}
\]

The recovery of the velocity and time derivative replaced with their complex counterparts \( V \) and \( \frac{d}{dt} \) indicates that this replacement implements the transition from the usually assumed differentiable geometry to a non-differentiable geometry with \( \frac{d}{dt} \) playing the role of a scale-covariant derivative. In the next section, we explore the implications of this transition in the case of one of the simplest mechanical systems: the harmonic oscillator.

5 Application to the one-dimensional harmonic oscillator

The fundamental relation of dynamics obtained above has both real and imaginary parts, which we can write separately, replacing \( \frac{d}{dt} \) and \( V \) by their expressions.

\[
\frac{\partial}{\partial t}V = D \Delta U + (V \cdot \nabla)V - (U \cdot \nabla)U = -\frac{1}{m} \nabla \Phi
\]

\[
\frac{\partial}{\partial t}U + D \Delta V + (V \cdot \nabla)U + (U \cdot \nabla)V = 0
\]

This system of differential equations is the same as Equations (34) in E. Nelson’s article entitled “Derivation of the Schrödinger equation from Newtonian Mechanics” [3]. However, they originate quite differently. In Nelson’s stochastic quantization, these equations result from the additional hypothesis of some underlying Brownian motion, which, in the quantum mechanical context, is characterized by a diffusion coefficient \( D = \frac{\hbar}{2m} \). In the Scale Relativity approach followed here, these equations result from the abandonment of
the usually implicit differentiability hypothesis with the choice \( \langle db_i \cdot db_j \rangle = 2D \delta_{i,j} dt \).

We may consider differentiable and classical solutions for which the kink velocity \( U = 0 \). The first equation then appears as the usual fundamental relation of dynamics \( \frac{dV}{dt} = -\frac{1}{m} \nabla \Phi \) and the second becomes \( \Delta V = 0 \) which is already ensured by the first one as \( V \) no longer can be an explicit function of position \( x \).

Inversely, we may concentrate on a new type of stationary motion characterized by \( \langle V \rangle = 0 \). It emerges entirely as a consequence of the non-differentiability. Among all the possible paths with this property, in order to simplify the above system of differential equations, we may choose a subset such that \( V = 0 \): \[ D \Delta U + (U \cdot \nabla)U = \frac{1}{m} \nabla \Phi \]
\[ \frac{\partial}{\partial t} U = 0 \]

The second equation implies that \( U \) depends only on \( x \). The solution \( U(x) \) to the first equation can be used in a time forward Langevin equation. Indeed, since we chose \( V = 0 \), we have \( v_+ = -v_- \), and \( U = v_+ \). Consequently, Equation ?? becomes
\[ dx_+ = U(x) dt + dB_+ \]
where \( dB_+ \) is a stochastic function such that \( \langle dB_+ \rangle = 0 \) and \( \langle dB_+ \cdot dB_j \rangle = 2D \delta_{i,j} dt \). Considering a finite time step, a natural choice for \( dB_+ \) is a Gaussian deviate of zero mean and with a standard deviation \( \sqrt{2D} dt \).

McClendon and Rabitz [?] simulated several quantum systems using the differential equations of Nelson’s stochastic quantization as a starting point [?]. The case of an infinite square well has been studied by Hermann [?] with the scale relativity approach presented here. Even more recently, the finite square well was also studied by Al-Rashid et al.[?]. Nottale [?] also simulated Young one and two-slit experiments as well as the hydrogen atom. Here, we consider the case of a one-dimensional harmonic oscillator for which \( \Phi(X) = \frac{1}{2} m \Omega^2 X^2 \), where \( m \) is the mass of the particle and \( \Omega \) is the frequency of the oscillator.

In one dimension, the differential equation for \( U \) becomes: \( D \frac{dU}{dx^2} + u \frac{dU}{dx} = \frac{4}{\Omega^2} (D \frac{dU}{dx} + \frac{1}{2} U^2) = \Omega^2 X \) and integrating once, \( D \frac{dU}{dx} + \frac{1}{2} U^2 + C = \frac{1}{2} \Omega^2 X^2 \) where \( C \) is an integration constant with the dimension of the square of a velocity. Introducing the dimensionless variables \( u = \frac{U}{\sqrt{2m\Omega}} \), \( x = \sqrt{\frac{\Omega}{2D}} X \), and \( c = \frac{C}{\sqrt{2m\Omega}} \), this equation takes the form
\[ \frac{du}{dx} + u^2 + c = x^2 \]

An obvious solution is \( u(x) = \pm x \) corresponding to \( c = \mp 1 \). Up to a multiplicative dimensional constant, \( u \) is the kink velocity. The solution \( u(x) = +x \) corresponds to the path being systematically kinked outward, away from the center of the harmonic well. We reject it for being non-physical and retain \( u(x) = -x \).
Table 1 The first few solutions to the dimensionless differential equation for \( u \) in terms of \( x \).

<table>
<thead>
<tr>
<th>( n )</th>
<th>( u_n )</th>
<th>( c_n )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(-x)</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>(-\frac{2}{3}x - \frac{2}{5}x^3)</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>(-\frac{4}{5}x^2 - \frac{10}{12}x^4)</td>
<td>5</td>
</tr>
<tr>
<td>3</td>
<td>(-\frac{8}{3}x^3 - \frac{36}{12}x^5 + \frac{12}{12}x^7)</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>(-\frac{16}{5}x^4 - \frac{112}{20}x^6 + \frac{72}{12}x^8)</td>
<td>9</td>
</tr>
<tr>
<td>5</td>
<td>(-\frac{32}{5}x^5 - \frac{320}{60}x^7 + \frac{120}{12}x^9)</td>
<td>11</td>
</tr>
</tbody>
</table>

All the solutions should share this property of having the path kinked inward, toward the center of the well in proportion to the distance of the path, at least when \(|x| \to \infty\). For this reason, we have looked for solutions in the form of the ratio of two functions \( u(x) = P(x)/Q(x) \) such that \( \lim_{x \to \pm \infty} u(x) = -x \).

Inspired by the work of Hermann [?] on the infinite square well, we used Hermite polynomials as the denominator \( Q(x) = H_n(x) \). This allows for the path to be kinked away from the regions where the Hermite polynomials have a root. The solution \( u(x) = -x \) enters this pattern for \( n = 0 \) as \( H_0(x) = 1 \) is the only Hermite polynomial with no roots. The solutions obtained for \( n \) in the range from 0 to 5 are given in Table ?? along with the values of \( c_n \). A general expression in terms of \( n \) has been obtained by S. Al-Rashid [?].

The values of \( c_n = 2n + 1 \) follow the same patterns as the eigenenergies of a quantum harmonic oscillator. In fact \( E_n = mC_n = 2mD\Omega^2(n + \frac{1}{2}) \) corresponds to the eigenenergies of the quantum harmonic oscillator for the identification \( \hbar = 2mD \).

We do not prove that all the solutions are such that \( c_n = 2n + 1 \). However we can observe how the solutions change when the integration constant \( c \) departs from \( c_n \). We write \( c = c_n + \delta c \) and \( u = u_n + y \). The differential equation gives: \( \frac{dy}{dx} + y^2 + 2u_n y + \delta c = 0 \). Far from the well center, \(|x| \gg 1\) and \( u_n = x \) and we can consider \(|y| \ll |u_n|\) so the \( y^2 \) term can be dropped. With this simplification and writing \( y = f(x)e^{x^2} \), it comes \( \frac{df}{dx} = -\delta Ce^{-x^2} \) and \( f(x) = -\frac{\delta c}{2} \sqrt{\pi} \text{sign}(x) e^{x^2} \) so that \( u(x) \approx -x - \frac{\delta c}{2} \sqrt{\pi} \text{sign}(x)e^{x^2} \). In the Langevin equation, the corrective term will be responsible for the path to be either kinked away from the center of the well or kinked toward the center of the well depending on the sign of \( \delta c \). The path is either forced toward the center or escapes indefinitely, in both cases departing from our requirement that \( \langle V \rangle = 0 \). This, in itself, is reminding of the fact that when solving the time independent Schrödinger equation for the harmonic well, the wave functions can be normalized only for the eigenenergies.

We numerically integrated the Langevin equation with \( \Omega = 0.001dt^{-1} \) while using \( \sqrt{2D} \) as a distance unit. Figure ?? shows the simulated path in a harmonic oscillator in mode \( n = 2 \). In panel (a), the path appears to spend
as much time on either side of the well and is rarely found close to the center of the well. The center of the well appears as a node separating two lobes. The symmetry degrades as shorter time intervals are considered in panels (b), (c) and (d). When shorter time intervals are considered, the path is increasingly likely to be found either in one lobe or another. When the path is in the region corresponding to one lobe, in order to migrate to a different lobe, it needs to undergo a large enough hop. Because of the Gauss distribution of the hops, large ones are very infrequent and the path spends a varying amount of time in each lobe.

Figure 3 shows the histograms of the positions of the path for \( n \) ranging from 0 to 5. Even and odd values of \( n \) were separated in the upper and lower panels respectively to improve visibility. For each value of \( n \), 20 independent numerical experiments were performed over \( 10^7 \) unit time steps \( dt \), each starting from a randomly chosen point. The red curves represent the histograms, which are compared to the Hermite functions, solutions of the Schrödinger equation, represented by the blue curves. The unevenness of the lobes of the red curves results from statistical fluctuations discussed above. The unevenness of the lobes is reduced as the duration of the simulation is increased.

The fact that these simulations with \( V = 0 \) reproduce the familiar solutions of the time independent Schrödinger equation is suggestive of quantum mechanics being a manifestation of the non-differentiability of the paths. This result depends on the very specific choice \( \langle db_i \pm db_j \pm \rangle = 2D \delta_{ij} dt \) we have made for the statistical property the stochastic term in the path representation (See Equation ??). It should be stressed however that these paths, simulated with a finite time step, are nothing more than a sampling of geometrical points along a non-differentiable geodesic, one in an infinite set which may constitute the
state of the quantum particle. Such simulated paths should not be regarded in any way as an actual trajectory followed by a quantum particle. This is further discussed in the next section.

Fig. 4 The histograms of the position of the harmonic oscillator with $\omega = 0.001$ followed in 20 numerical experiments of $10^7$ steps for $n=0, 2$ and 4 (top) and $n=1, 3$ and 5 (bottom) are shown by the red dashed curves. The solid blue curves represent the Hermite functions, solution of the Schrödinger equation for the same harmonic oscillator.

6 Conclusion and discussion

In Section ??, we have exposed an approach to represent non-differentiable paths with two terms (Equation ??). One describes the usual displacements revealed by inspection at some given scale and the other is the residual to be revealed at finer resolution. We were led to recognize the velocity to be double-valued, which can conveniently be expressed by making use of complex numbers. Correspondingly, we defined a complex time-differential operator
(Equation ??). Considering a regular field, the complex time-differential operator acting on the field takes the form of a covariant differential such as in Equation ?? where the stochastic part of the path representation is chosen to be a Wiener process characterized by a diffusion constant $\mathcal{D}$.

In Section ??, applying the stationary action principle, we established that the dynamics of non-differential path is obtained by replacing the usual time derivative by the complex time-differential operator. In particular, we obtain a generalization of Newton’s fundamental relation of dynamics in Equation ??.

Finally, in Section ??, we saw that, in the case of a zero average velocity, the fundamental relation of dynamics take the form of a Langevin equation, which we numerically integrated in the case of a simple harmonic oscillator. We observed that the statistical distribution of the position of the path in the course of the numeric simulation reproduces the square of the magnitude of the solution of the time independent Schrödinger equation. The quantum-like behavior appears to manifest itself as a consequence of the non-differentiability of the paths for the specific choice we made for the statistics of the stochastic term in the path representation. The resulting scaling properties of the non-differentiable path are expressed by the last term of the scale-covariant time-differential operator and are responsible for the quantum-like behavior. This is very similar to the situation in general relativity where the curved property of space time is expressed by the affine connection in the covariant derivative and is responsible for the gravitation phenomena.

Figure ?? deserves some reflection. Let us imagine that in-between time steps, a physical particle travels along a differentiable trajectory, in such a way that any finer resolution would not reveal any new structures in the path. Then, from panel (a) to panel (d) in Figure ??, we observe the progressive transition from a fractal path at large time scales toward a differentiable path at shorter time scale. If the simulation had been carried out for the same time duration but with a time step 1000 times smaller, then the right most panel (d) would look like the leftmost panel (a). Meanwhile, in panel (a), the number of hopping from one side to the other, would be so great that they would be indistinguishable and for all practical purposes in that graph, at any given time, the particle could only be probabilistically described as being on one side or the other. Pushing this even further by making the elementary time step tend to zero, all four panels of Figure ?? would have the exact same appearance as we would be infinitely far away from the scale at which the transition from probabilistic behavior to trajectory-like behavior may take place. Even though we have been thinking about paths in the usual sense, in the limit of infinitesimal time steps, the notion of position loses its meaning. We do not have one path anymore but all of them at once and the question of the position of the particle can only be answered statistically. The collection of all the paths can be thought of as a fluid whose density sets the chance probability to observe the particle in a given range of positions at a given time. It is that entire set that may constitute the actual state of the quantum particle.
The follow up paper [?] will establish Schrödinger’s equation as a reformulation of Newton’s generalized relation of dynamics (Equation ???) and will provide a discussion of the axioms of standard quantum mechanics in the context of Scale Relativity. Standard quantum mechanics is observed at small scales, below the de Broglie wavelength, where paths have fractal dimensions \( D_F = 2 \) while classical mechanics is recovered at larger scale where paths return to having fractal dimension \( D_F = 1 \). However, we have seen that the present development does not depend on the non-differentiable and fractal nature \( (D_F = 2) \) of the paths to be maintained down to infinitesimal scales. This implies that macroscopic systems in which trajectories may be differentiable below some scale and non-differentiable with \( D_F = 2 \) only at some large enough scales can be expected to display quantum-like behavior provided the Scale Relativity principle applies.

Acknowledgements  The authors are grateful to Eugene Mishchenko, Patrick Fleury, and Janvida Rou for interesting discussions and their helpful comments and suggestions to clarify the text.

References