Quick Note: Web Assign had the wrong answers for parts A and C of this problem and generated confusion for many students. This document will clear up that confusion or else it gets the hose again. We’ll solve the problem using two different approaches: analysis of the system as a whole and then analysis of the wheel.

1 The Question

A counterweight of mass $m$ is attached to a light cord which is wound around a pulley as shown in the figure below. The pulley is a thin hoop of radius $R$ and mass $M$. The spokes have negligible mass.

- What is the net torque on the system about the axle of the pulley?

- When the counterweight has a speed $v$, the pulley has an angular speed $\omega = \frac{v}{R}$. Determine the magnitude of the total angular momentum of the system about the axle of the pulley.

- Using your result from (b) and $\vec{\tau} = \frac{d\vec{L}}{dt}$, calculate the acceleration of the counterweight.

1.1 The Figure
2 Solution Method I: System as a Whole

This approach was the original intent of the problem (but the answers to Web Assign did not correspond to this approach). Essentially, we solve the problem by analyzing the system as a whole (wheel + counterweight).

2.1 The Net Torque of the System

To determine the net torque on the system about the axle of the pulley, we must find the torque on both the wheel and the counterweight:

$$\vec{\tau}_n = \vec{\tau}_w + \vec{\tau}_c$$  \hspace{1cm} (1)

Each of these torques is generated by a force and we’ll determine each of these torques using the vector cross product:

$$\vec{\tau} = \vec{r} \times \vec{F}$$  \hspace{1cm} (2)

For the wheel torque, let’s use this diagram to better understand the two vectors in our equation.

![Diagram of wheel and force vectors]

The $\vec{r}$ and $\vec{F}$ vectors for the wheel are then:

$$\vec{r} = -R\hat{i} \hspace{1cm} \vec{F} = -T\hat{j}$$  \hspace{1cm} (3)

The torque for the wheel is:

$$\vec{\tau}_w = \vec{r} \times \vec{F} = (-R\hat{i}) \times (-T\hat{j})$$  \hspace{1cm} (4)

And this computes to:

$$\vec{\tau}_w = RT\hat{k}$$  \hspace{1cm} (5)

For the counterweight torque, let’s use this diagram to better understand the two vectors in our equation.

![Diagram of counterweight and force vectors]
The $\mathbf{r}$ and $\mathbf{F}$ vectors for the wheel are then:

$$\mathbf{r} = -R \hat{i} - Q \hat{j} \quad \quad \mathbf{F} = (T - mg) \hat{j}$$ (6)

Note that to determine $\mathbf{r}$, we’re simply writing down how far in the x and y directions we’d have to walk to reach the location of the counterweight. We don’t know the location of the block in the y-direction, so I’m just calling it some unknown value $Q$ for now.

The torque for the counterweight is:

$$\mathbf{\tau}_c = \mathbf{r} \times \mathbf{F} = (-R \hat{i} - Q \hat{j}) \times ((T - mg) \hat{j})$$ (7)

And this computes to:

$$\mathbf{\tau}_c = (-RT + mgR) \hat{k}$$ (8)

Now that we know the torques for the wheel and the counterweight, we can find the net torque on the whole system by simply adding them together.

$$\mathbf{\tau}_n = \mathbf{\tau}_w + \mathbf{\tau}_c = mgR \hat{k}$$ (9)

### 2.2 Total Angular Momentum

The total angular momentum of the system will be the sum of the angular momentum of the two objects (wheel and counterweight):

$$\mathbf{L}_t = \mathbf{L}_w + \mathbf{L}_c$$ (10)

The quick way to determine the angular momentum of the wheel is to use:

$$\mathbf{L} = I \omega$$ (11)

For the wheel spinning around its center of mass, this becomes:

$$\mathbf{L}_w = MR^2 \omega_w$$ (12)

But, since $\omega = \omega_w / R$, the momentum reduces to:

$$\mathbf{L}_w = MR \omega$$ (13)

Now, let’s get the angular momentum for the counterweight. The definition of angular momentum is a vector cross product:

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$ (14)

For the counterweight, this becomes:

$$\mathbf{L}_c = (-R \hat{i} - Q \hat{j}) \times (-mv \hat{j})$$ (15)

This computes to:

$$\mathbf{L}_c = mRv \hat{k}$$ (16)

Therefore, the total angular momentum of this system is:

$$\mathbf{L}_t = \mathbf{L}_w + \mathbf{L}_c = (m + M)Rv \hat{k}$$ (17)
2.3 The Acceleration of the Counterweight

We are instructed to take a time derivative of the angular momentum to help us find the acceleration of this system. This time rate of change of the angular momentum is equivalent to the net torque on the system.

\[ \tau_n = \frac{dL}{dt} \]  

(18)

Using our expression for torque from Part (A) and our expression for angular momentum from Part (B), this equation becomes:

\[ mgR = (m + M)R\frac{dv}{dt} \]  

(19)

But, the time rate of change of the velocity is the acceleration! Thus,

\[ mgR = (m + M)Ra \]  

(20)

Rearranging this expression for acceleration yields:

\[ a = \frac{mg}{m + M} \]  

(21)

Note: The reason many of you were getting Part (C) incorrect was because it required the torque from Part (A).

3 Solution Method II: Analysis of the Wheel

In this approach, we will focus our efforts almost exclusively on the wheel. I feel that this approach is more intuitive because it has obvious cause-and-effect relationships between forces and rotations.

3.1 The Net Torque on the Wheel

Here, we’re ignoring the pleas of Web Assign to calculate the net torque of the system. Instead, we’re just going to focus on the net torque on the wheel. Only one force acts on the wheel that can create a torque and that force is the tension in the cord. Refer back to the diagrams from earlier in this document to produce the vectors \( \vec{r} \) and \( \vec{F} \).

\[ \tau_w = \vec{r} \times \vec{F} = (-R\hat{i}) \times (-T\hat{j}) \]  

(22)

\[ \tau_w = RT\hat{k} \]  

(23)

We have an unknown in our torque equation: the tension in the cord. To find the tension, we’ll first perform a Newton’s 2nd law analysis on the counterweight.

\[ \sum F = ma \]  

(24)

Choosing up to be the positive direction, yields:

\[ T - mg = -ma \]  

(25)

This equation has two unknowns (\( T \) and \( a \)). So, we turn to the torque equation:

\[ \sum \tau = I\alpha \]  

(26)
This equation has great physical relevance here. The equation is saying whatever torques act on my wheel will, in turn, create a rotation for the wheel. In this particular example, the tension in the cord is the force generating the torque on the wheel.

For the wheel, this equation becomes:

\[ RT = MR^2 \alpha \]  \hspace{1cm} (27)

But, since \( \alpha = \frac{a}{R} \), we get:

\[ RT = MR^2 \frac{a}{R} \]  \hspace{1cm} (28)

This reduces to:

\[ T = Ma \]  \hspace{1cm} (29)

Now we have a system of two equations (Eq 25 and Eq 29) and two unknowns (\( T \) and \( a \)). Solving this system for the tension, \( T \), allows us to then find the net torque acting on the wheel. The tension is determined to be:

\[ T = \frac{mMg}{m + M} \]  \hspace{1cm} (30)

Thus, the net torque acting on the wheel is found to be:

\[ \tau_w = \frac{mMgR}{m + M} \]  \hspace{1cm} (31)

3.2 Total Angular Momentum

You would proceed here just as you in Section 2 of this document. I’d argue that this step isn’t needed at all to determine the acceleration of your counterweight.

3.3 The Acceleration of the Counterweight

This is the main goal of the problem - to determine the acceleration of the counterweight. You’ve already done most of the work for this step. Notice that you have expressions for tension in Equations 29 and 30. Let’s set them equal to each other:

\[ Ma = \frac{mMg}{m + M} \]  \hspace{1cm} (32)

Solving this equation for the acceleration yields:

\[ a = \frac{mg}{m + M} \]  \hspace{1cm} (33)

This is exactly the same result you get by employing all the machinery needed to analyze the entire system instead of just analyzing the wheel.