The calculation of a force acting on a metal shell is illustrated below (see also Chapter 2.5):

\[ \vec{E}_0 \]

\[ \vec{E}_2 = \vec{E}_0 + \frac{\vec{B}}{2\varepsilon_0} \]

\[ \vec{E}_1 = \vec{E}_0 - \frac{\vec{B}}{2\varepsilon_0} \]

\[ \vec{E}_0 = \frac{\vec{E}_1 + \vec{E}_2}{2} \text{ — external field} \]

\[ \delta F = \vec{E}_0 \cdot \delta S = \frac{\vec{E}_1 + \vec{E}_2}{2} \cdot \delta S \]

In case of a spherical shell, \( \vec{E}_1 = 0 \)

\[ \delta F = \frac{Q}{8\pi \varepsilon_0 R^2} \cdot \frac{Q}{4\pi R^2} \cdot \vec{n} \cdot \delta S \]

The net force is along the x-direction:

\[ \delta F = \frac{Q^2}{8\pi \varepsilon_0 R^2} \frac{\delta S \cos \theta}{4\pi R^2} \]

By noting that \( \Sigma \delta S \cos \theta = \pi R^2 \) (the area of a circle of radius \( R \)) we find

\[ F_x = \Sigma \delta F_x = \frac{Q^2}{8\pi \varepsilon_0 R^2} \frac{\pi R^2}{4\pi R^2} = \frac{Q^2}{32\pi \varepsilon_0 R^2} \]
Electric field has been found in

Homework 2, problem 4:

\[ E_z = \frac{Q}{2\pi \varepsilon_0 R^2} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \approx \begin{cases} \frac{Q}{z^2 \varepsilon_0 R^2}, & z \ll R \\ \frac{Q}{4\pi \varepsilon_0 z^2}, & z \gg R \end{cases} \]

To find magnetic field we first solve an auxiliary problem of a circular loop of radius \( S \) with current \( I \).

\[ \mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi} \int \frac{\mathbf{dl} \times (\mathbf{R} - \mathbf{r})}{|\mathbf{R} - \mathbf{r}|^3} \]

Biot-Savart law gives that each element \( dl \) contributes

\[ \frac{\mu_0 I}{4\pi} \frac{dl}{S^2 + z^2} \frac{S}{\sqrt{S^2 + z^2}} \cos \theta \]

to the total \( \mathbf{B} = (0, 0, B_z) \) field. Since \( \int dl = 2\pi S \) we get

\[ B_z = \frac{\mu_0 IS^2}{2(S^2 + z^2)^{3/2}} \]

To find magnetic field of a rotating disk we split it into thin rings with thickness \( ds \) each carrying current

\[ dI = \frac{Q}{4\pi R^2} \cdot \frac{SR}{\text{velocity}} \cdot ds \]

charge density
Using the result of the auxiliary problem:

\[ dB_z = \frac{\mu_0 S^2}{2(\sqrt{S^2 + z^2})^{3/2}} \frac{QR}{\pi R^2} \frac{s'ds}{s'} \]

and integrating over \( ds \) from 0 to \( r \):

\[ B_z(z) = \frac{\mu_0 QR}{2\pi R^2} \int_{0}^{r} \frac{s'^3 ds'}{(S^2 + z^2)^{3/2}} \]

The integral is taken by the substitution

\[ S = z \tan \theta_0 \quad \text{and} \quad ds = \frac{z d\theta_0}{\cos^2 \theta_0} \]

\[ \int_{0}^{r} \frac{s'^3 ds'}{(S^2 + z^2)^{3/2}} = z \int_{0}^{\theta_0} \frac{\sin^3 \theta_0}{\cos^2 \theta_0} d\theta_0 = z \int_{0}^{1} \frac{1 - \cos^2 \theta_0 d\theta_0}{\cos^2 \theta_0} = \frac{\tan \theta_0}{\cos \theta_0} \]

\[ \frac{\tan \theta_0}{\cos \theta_0} = \frac{z^2}{\sqrt{z^2 + 2z}} = \frac{z^2 + R^2}{\sqrt{2z + R^2}} - z \]

The answer

\[ B_z(z) = \frac{\mu_0 QR}{2\pi R^2} \left( \frac{z^2 + R^2}{\sqrt{2z + R^2}} - z \right) \]

For \( z \ll R \) \( B_z(z) \approx \frac{\mu_0 QR}{2\pi R^2} \)

For \( z \gg R \) we expand \( \frac{1}{\sqrt{2z + R^2}} = \frac{1}{z} \left( 1 - \frac{R^2}{2z^2} + \frac{3R^4}{8z^4} \right) \)

and obtain \( B_z(z \gg R) \approx \frac{\mu_0 QR^2}{8\pi z^3} \)

Note that this is simply a field of a magnetic dipole \( \mu = \frac{QR^2z}{4} \), in agreement with
Problem 3a from homework 7.

Since everywhere on the axis \( \vec{B} \parallel \vec{E} \), energy flux is zero.

Within the plane of the disk \( S \) rotates in the same direction as the disk does.

\[
\vec{S} \propto \vec{E} \times \vec{B}
\]

Field line picture near the end of a solenoid:

last coil (upper view)

\[
\vec{F} \propto \vec{E} \times \vec{B}_L \quad \text{downwards}
\]

Magnetic force acts to compress the solenoid.

Magnetic field

\[
B = \mu_0 \frac{N}{L} I
\]

Magnetic energy

\[
E_m = \pi R^2 \frac{B^2}{2\mu_0} = \frac{\mu_0 N^2 I^2 R^2}{2L}
\]
When the length is increased \( L \rightarrow L + \Delta L \)

\[
\delta E_m = -\frac{\mu_0 N I^2 \frac{\pi R^2}{2L^2}}{2L^2} \Delta L
\]

magnetic energy actually decreases.

Why is it then energetically not favorable for the solenoid to expand?

Because the Faraday EMF must perform additional work. Flux change

\[
\delta \Phi = \frac{\pi R^2}{2} \delta B = -\frac{\pi R^2}{L^2} I \delta L
\]

when \( \delta L > 0 \) \( \delta \Phi < 0 \) and Faraday EMF works towards increasing the magnetic field (or flux – Lenz’s rule)

Work:

\[
\delta W = -\frac{\delta \Phi}{\delta t} I \Delta t = \frac{\pi R^2 \mu_0 N I^2}{L^2} \delta L
\]

Note that \( \delta W \) is twice as much as the decrease in magnetic energy. The length increase is energetically unfavorable without external force.

Such a force \( F \) has to perform work \( F \Delta L \). This work goes into the change of magnetic energy and work performed on charges:

\[
F \Delta L = \Delta E_m + \Delta W = \frac{1}{2} \Delta W
\]
Thus \[ F_{\text{ext}} = \frac{1}{2} \frac{\Delta W}{\Delta L} = \frac{M_0 \pi R_0^2 N_0^2 L^2}{2L^2} \]

For the values given in the problem

\[ F_{\text{ext}} = \frac{4\pi \cdot 10^{-7} \frac{N}{A^2} \pi (0.05m)^2 (100)^2 (5A)^2}{2 \cdot (0.1m)^2} \approx 0.12N \]

(4) Similarly to Problem in HW13, the instant energy loss due to dipole radiation is

\[
\frac{dw}{dt} = \frac{M_0 \cdot \frac{d^2}{dt^2}}{6\pi c} \quad d = q \cdot r
\]

\[ d \text{ is acceleration } r \text{ is }
\]

\[ \text{found from the combination of the 2nd Newton's law and Coulomb's law:}
\]

\[
m \ddot{r} = -\frac{Qq}{4\pi \varepsilon_0 r^2} \quad \ddot{d} = q \cdot \ddot{r} = -\frac{Qq^2}{4\pi \varepsilon_0 m r^2}
\]

Thus,

\[
\frac{dw}{dt} = \frac{M_0 \cdot \frac{Q^2 q^2}{96 \pi \varepsilon_0 c R^4 m^2}}{\frac{dw}{dr} \frac{dr}{dt} = \frac{dw}{dr} \dot{r}}
\]

Velocity \( \dot{r} \) can be expressed via \( r \) by means of the energy conservation:

\[
\frac{m \dot{r}^2}{2} = \frac{Qq}{4\pi \varepsilon_0 r} \Rightarrow \dot{r} = \sqrt{\frac{Qq}{2\pi \varepsilon_0 m r}}
\]

Thus
\[ \frac{dw}{dr} = \frac{\mu_0 Q^2 q^4}{8 \pi^2 \epsilon_0^2 c r^4} \frac{\sqrt{2 \mu m \sigma_0}}{V a q \ m^2} \]

Integrating now this equation from \( r \) to \( \infty \), we find the total energy loss:

\[ W = \frac{\mu_0 Q^{3/2} q^{7/2}}{48 \pi^2 \epsilon_0^{3/2} m^{3/2} c} \int_{r}^{\infty} \frac{dr}{r^{3/2}} \]

\[ = -\frac{\mu_0 Q^{3/2} q^{7/2}}{120 \pi^2 \epsilon_0^{5/2} m^{3/2} c R^{5/2}} \]