Homework 5

Problem 1. We look for a solution in the form (which is a solution to Laplace equation!)

\[ \psi_{r<R} = -E_r \cos \theta \]

\[ \psi_{r>R} = -E_r \cos \theta + \frac{B \cos \theta}{r} \]

Boundary conditions require

a) continuity of tangential electric field at the interface (i.e. continuity of \( \psi \)):

\[ -E_R = -E_0 R + \frac{B}{R} \]

b) continuity of \( D_n \) at the interface

\[ -\varepsilon \left. \frac{\partial \psi_{r<R}}{\partial r} \right|_{r \to R} = -\varepsilon_0 \left. \frac{\partial \psi_{r>R}}{\partial r} \right|_{r \to R} \]

which gives

\[ \varepsilon E = \varepsilon_0 E_0 + \varepsilon_0 \frac{B}{R^2} \]

Excluding \( B \):

\[ \varepsilon E = \varepsilon_0 E_0 + \varepsilon_0 (E_0 - E) \]

and

\[ E = \frac{2 \varepsilon_0}{\varepsilon + \varepsilon_0} E_0 \]

Since \( \varepsilon = \varepsilon_0 (1 + \chi) \) we can write, recovering now the vector notations:

\[ \vec{E} = \frac{2}{2 + \chi} \vec{E}_0 \]
Problem 2. a-b) Denoting the linear charge density created by the source on the wire \( \lambda \) (which is free charge!), we can apply Gauss law for \( \Phi_0 = \rho_{\text{free}} \) in its integral form:
\[
\oint \Phi_0 \cdot d\mathbf{a} = \Phi_{\text{free}}
\]
to a Gaussian cylinder of length \( L \) and radius \( R \) \( (R_0 < R < R_2) \):
\[
2\pi r L \Phi_0(r) = L \lambda \quad \Rightarrow \quad \Phi_0(r) = \frac{\lambda}{2\pi \lambda}
\]
Using \( \Phi = \varepsilon_0 \mathbf{E} \), we can write:
\[
\mathbf{E}(r) = \frac{\lambda}{2\pi \varepsilon_0 r}, \quad R_0 < r < R_1
\]
\[
\mathbf{E}(r) = \frac{\lambda}{2\pi \varepsilon_1 r}, \quad R_1 < r < R_2
\]
\( \mathbf{E}(r) = 0 \) elsewhere.

We can now find \( \lambda \) by relating it to a voltage drop:
\[
\int_{R_0}^{R_2} \mathbf{E}(r) \cdot d\mathbf{r} = V \quad \Rightarrow \quad \lambda \int_{R_0}^{R_1} \frac{dr}{r} + \lambda \int_{R_1}^{R_2} \frac{dr}{r} = V
\]
and
\[
\lambda = \frac{2\pi V}{\varepsilon_1 \frac{R_1}{R_0} + \varepsilon_2 \frac{R_2}{R_1}}
\]
(Note that this expression determines...
c) Polarization is found from
\[ \vec{D} = \varepsilon_0 \vec{E} + \vec{P} \Rightarrow \vec{P} = \vec{D} - \varepsilon_0 \vec{E} = (\varepsilon - \varepsilon_0) \vec{E} \]
\[ \vec{P}(\vec{r}) = \frac{(\varepsilon_1 - \varepsilon_0) \lambda}{2\pi \varepsilon_1 r}, \quad R_0 < r < R_1 \]
\[ \vec{P}(\vec{r}) = \frac{\varepsilon_2 - \varepsilon_0) \lambda}{2\pi \varepsilon_2 r}, \quad R_1 < r < R_2. \]

d) Bulk bound charges are found from
\[ \rho = -\nabla \cdot \vec{P} \]
In our (axially symmetric) case
\[ \nabla \cdot \vec{P} = \frac{1}{r} \frac{\partial}{\partial r}(r \vec{P}_r) = 0 \]
and we get that all bulk bound charges are zero!

Surface bound charges are determined by \( \vec{Z} = \vec{P}_n \) at the interfaces:
At \( r = R_0 \), \( Z_0 = \frac{(\varepsilon_1 - \varepsilon_0) \lambda}{2\pi \varepsilon_1 R_0} \)
Note "-" sign — outside normal direction is inwards!
At \( r = R_2 \), similarly \( Z_2 = \frac{(\varepsilon_2 - \varepsilon_0) \lambda}{2\pi \varepsilon_2 R_2} \)
Finally, at \( r = R_1 \), there are two surface bound charges (from two adjacent dielectrics)
\[ Z_1 = \frac{(\varepsilon_1 - \varepsilon_0) \lambda}{2\pi \varepsilon_1 R_1} - \frac{(\varepsilon_2 - \varepsilon_0) \lambda}{2\pi \varepsilon_2 R_1} \]

One can easily verify that the net charge is zero, i.e.

\[ 2\pi (R_0 Z_0 + R_1 Z_1 + R_2 Z_2) = 0 \]