(1) a) A circular loop of radius $r < a$ encloses no current, thus $B(r < a) = 0$.

For $r > a$ the current is simply $I$, so

$$B \cdot 2\pi r = \mu_0 I \implies B(r > a) = \frac{\mu_0 I}{2\pi r}$$

b) the field outside is the same as in a).

Inside the wire, the density of current is $j(r) = cr$, where the constant $c$ is found from

$$c \int_0^a 2\pi r \, dr \cdot r = I,$$

which gives

$$c = \frac{3I}{2\pi a^3} \Rightarrow j(r) = \frac{3Ir}{2\pi a^3}$$

The Ampere law now gives

$$B \cdot 2\pi r = \mu_0 \int_0^r 2\pi r \, j(r) = \frac{3\mu_0 I r}{a^3} \int_0^r 2\pi r \, dr = \frac{\mu_0 IR^3}{a^3}$$

so that $B(r) = \frac{\mu_0 IR^2}{2\pi a^3}$, and the fields match each other at the surface of a cylinder.

(2) Magnetic field is given by curl of Eq. (5.83):

$$\mathbf{B}(r) = \frac{\mu_0}{4\pi} \nabla \times \frac{\mathbf{m} \times \mathbf{r}}{r^3}$$
Using Eq. (vi) from page 21 and using that \( \overrightarrow{\mathbf{r}} \) does not depend on \( \overrightarrow{\mathbf{r}} \) we write:

\[
\nabla \times \frac{\overrightarrow{\mathbf{m}} \times \overrightarrow{\mathbf{r}}}{r^3} = -(\overrightarrow{\mathbf{m}} \cdot \nabla) \frac{\overrightarrow{\mathbf{r}}}{r^3} + \overrightarrow{\mathbf{m}} \left( \nabla \cdot \frac{\overrightarrow{\mathbf{r}}}{r^3} \right)
\]

The second term vanishes for any \( \overrightarrow{\mathbf{r}} \) similar to the vanishing of a divergence of electric field of a point charge. The first term gives:

\[
(\overrightarrow{\mathbf{m}} \cdot \nabla) \frac{\overrightarrow{\mathbf{r}}}{r^3} = (m_x \frac{\partial}{\partial x} + m_y \frac{\partial}{\partial y} + m_z \frac{\partial}{\partial z}) \left( \frac{x}{r^3}, \frac{y}{r^3}, \frac{z}{r^3} \right)
\]

The \( x \)-component is found as follows:

\[
m_x \frac{\partial}{\partial x} \frac{x}{r^3} = \frac{m_x}{r^3} - \frac{3m_x x^2}{r^5}
\]

\[
m_y \frac{\partial}{\partial y} \frac{x}{r^3} = \frac{3m_y xy}{r^5}
\]

\[
m_z \frac{\partial}{\partial z} \frac{x}{r^3} = \frac{3m_z xz}{r^5}
\]

or:

\[
(\overrightarrow{\mathbf{m}} \cdot \nabla) \frac{\overrightarrow{\mathbf{r}}}{r^3} = \frac{m_x}{r^3} - \frac{3x(m_x x + m_y y + m_z z)}{r^5}
\]

Similar calculations are performed for \( y \) and \( z \)-component. They all are summarized by:

\[
(\overrightarrow{\mathbf{m}} \cdot \nabla) \frac{\overrightarrow{\mathbf{r}}}{r^3} = \frac{\overrightarrow{\mathbf{m}}}{r^3} - \frac{3 \overrightarrow{\mathbf{r}} (\overrightarrow{\mathbf{m}} \cdot \overrightarrow{\mathbf{r}})}{r^5}
\]

So finally we get:

\[
\overrightarrow{\mathbf{B}} (\overrightarrow{\mathbf{r}}) = \frac{\mu_0}{4\pi} \left( \frac{3 \overrightarrow{\mathbf{r}} (\overrightarrow{\mathbf{m}} \cdot \overrightarrow{\mathbf{r}}) - \overrightarrow{\mathbf{m}}}{r^5} \right)
\]

\( \Box \) Auxilliary problem: a hoop of radius \( r \) is charged with linear density \( \lambda \) and spinning with
Frequency $\omega$.

Area: $\pi r^2$

Current: $I = \frac{\Delta Q}{\Delta t} = \frac{\omega r \lambda}{\Delta t} = \omega r \lambda$

$\mu = \pi r^3 \omega \lambda$

a) We now split the disk into elementary ‘hoops’ with the width $dr$.

Surface density $\zeta = \frac{Q}{\pi R^2}$

Linear density is $d\lambda$

$2\pi r d\lambda = \zeta \cdot da$, where $da = 2\pi r dr$

$d\lambda = \zeta dr = \frac{Q}{\pi R^2} dr$

The total magnetic moment is the sum of all ‘hoops’:

$\mu_{\text{disk}} = \int_0^R 2\pi r dr \cdot \pi r^2 \omega = \frac{2R^4}{4} \pi \omega = \frac{Q R^2}{4} \omega$

b) Similarly, we split a ball into elementary disks:

Volume charge density

$P = \frac{Q}{\frac{4\pi}{3} R^3} = \frac{3Q}{4\pi R^3}$

Surface charge $d\sigma$ of an elementary disk of thickness $dz$ is

$\pi (R^2 - z^2) dz \cdot \frac{3Q}{4\pi R^3} = d\sigma \Rightarrow d\sigma = \frac{3Q(R^2 - z^2)}{4R^3} dz$
The sum of magnetic moments is now (with the help of the result of part a)

\[ M_{\text{ball}} = \int_{-R}^{R} \frac{3Q(R^2-z^2)}{4R^3} dz \cdot (R^2-z^2) \frac{\omega}{4} = \]

\[ = \frac{3Q\omega}{8R^3} \int_{0}^{R} dz \cdot (R^2-z^2)^2 = \frac{3Q\omega}{8} \frac{R^2}{2} \left[ 1 - \frac{2}{3} + \frac{1}{5} \right] \]

\[ = \frac{2}{15} Q R^2 \omega \]

Note that \( \frac{2}{15} > \frac{2}{4} \) the magnetic moment of a disk is (almost twice) bigger.