

PMT Basic Electricity and Magnetism 3910 Multipoles

SPHERICAL COORDINATES, MULTIPOLE DECOMPOSITION, INTRODUCTION

Spherical coordinates r, θ, ϕ are the most useful coordinates for many problems that occur in electricity and magnetism. In this system the angle θ (the “polar angle” or “colatitude”) ranges from 0 to π , and ϕ (the “azimuthal angle”) ranges from 0 to 2π . These angles, with these ranges, characterize all possible directions from the origin. In analyzing such things as electric and magnetic fields we are often interested in functions of r, θ, ϕ . A typical example would be to find the electrostatic potential as a function of r, θ, ϕ .

Any function of r, θ, ϕ is of course a function of θ, ϕ at any particular value of r . What we will be doing in this chapter is to introduce a wonderful piece of mathematics. We will explain how a function that depends in an *arbitrary* way on variables θ, ϕ can be written as a sum of pieces with very *specific* patterns of dependence on θ, ϕ . These pieces are called the multipoles of the arbitrary function. To change points of view – from arbitrary dependence on θ, ϕ to parts with specific dependence – is called decomposing the (arbitrary) function into its multipoles.

For reasons of simplicity we will specialize; we will consider functions (e.g., the electrostatic potential) that are independent of ϕ . Such functions are said to be azimuthally symmetric. It turns out that we will avoid much complexity of detail by doing this, and not miss much of the meaning. Later we will briefly mention how to extend what we are doing here to the more general case of functions that are not azimuthally symmetric.

With our simplifying assumption, no ϕ dependence, we are concerned for the moment only with functions of θ . (We continue to ignore the dependence of physical fields on r ; the r dependence is certainly important, but it is best to put it aside for the moment.) Instead of thinking of θ as our basic variable, we will think of $\cos \theta$ as the basic variable. This is certainly allowed. If you are given a value of $\cos \theta$ in the range -1 to $+1$ you can find a unique corresponding value of θ in the range 0 to π , and vice versa. So we now turn from the consideration of functions $f(\theta)$ for $0 \leq \theta \leq \pi$ to $f(\cos \theta)$ with $-1 \leq \cos \theta \leq 1$. Since it is cumbersome to write our variable as “ $\cos \theta$,” we will simply denote it as “ x ,” and will consider functions $f(x)$ with $-1 \leq x \leq 1$.

We now go off on a mathematical side excursion to develop a few tools that we will need.

LEGENDRE POLYNOMIALS

Suppose you have some function $f(x)$, suppose that you are interested in the interval $-1 \leq x \leq 1$, and suppose you want to write your function $f(x)$ in some useful approximate way as a polynomial. There are good reasons that such an approximation would be useful. Polynomials are very easy for both people and computers to deal with. One way of doing it would be to use a Taylor expansion. For example, if the function is $f(x) = \exp(x^2 + x^3)$, we could approximate it with its Taylor series:

$$e^{x^2+x^3} \approx 1 + x^2 + x^3 + (1/2) x^4 . \quad (1)$$

This approximation is quite good for small values of x . A different kind of approximation, one that is pretty good over the whole interval from -1 to +1, is to build the function as a sum of special polynomials called Legendre polynomials, and universally written as $P_\ell(x)$. The first few of the Legendre polynomials are

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2 - 1) \\ P_3(x) &= \frac{1}{2}(5x^3 - 3x) \\ P_4(x) &= \frac{1}{8}(35x^4 - 30x^2 + 3) \\ P_5(x) &= \frac{1}{8}(63x^5 - 70x^3 + 15x) . \end{aligned}$$

The defining property of the Legendre polynomials is that each one is a solution of the following second order differential equation, called the Legendre equation:

$$\frac{d}{dx} \left((1-x^2) \frac{dP_\ell}{dx} \right) = -\ell(\ell+1)P_\ell , \quad (2)$$

and at $x = 1$ has the value

$$P_\ell(x = 1) = 1 \quad \text{for all } \ell . \quad (3)$$

Notice that the Legendre polynomials have “parity.” They are either even or odd functions of x . More specifically, when ℓ is an even integer, $P_\ell(x)$ is an even

function of x ; when ℓ is an odd integer, $P_\ell(x)$ is an odd function of x . In Maple the Legendre polynomials can be found by first entering `with(orthopoly)`. The command `P(2, x)` will then give you $(3/2)x^2 - 1/2$ and so forth.

An approximation with Legendre polynomials takes the form

$$f(x) \approx c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x) + \dots \quad (4)$$

For example, in place of the Taylor expansion in Eq. (1) the Legendre approximation, up to the $c_4 P_4(x)$ term, is

$$e^{x^2+x^3} \approx 1.054623007 - .602082380 x - 0.018218772 x^2 + 3.274718510 x^3 + 2.898786697 x^4 . \quad (5)$$

Very close to $x = 0$, this way of writing $\exp(x^2 + x^3)$ is not as good as the polynomial of the same order in Eq. (1). But the Legendre series approximation gives reasonable accuracy for the entire interval $[-1, +1]$. The success of the Legendre approximation, and a comparison with the Taylor series approximation is shown in Fig. 1. What is illustrated here is the ability of the Legendre polynomials to give a good approximation over the interval $[-1, +1]$. It is important to understand that this property is very specific to that interval. The Legendre polynomials are *not* particularly successful on a wider interval. Figure 2 shows what happens if we try to use the approximation in Eq. (5) for $x > 1$. (We could find another set of polynomials optimized, say, for usefulness on the interval $[-1, +1.5]$, but that would take us too far from the application of interest here.)

In order to use Legendre polynomials, of course, we need to know how to find the coefficients c_0, c_1, c_2, \dots in a series like Eq. (4). A major reason that approximations with Legendre series is so useful is that these coefficients are so easy to evaluate. It turns out (although we won't derive it) that

$$\int_{-1}^{+1} P_{\ell_1}(x) P_{\ell_2}(x) dx = 0 \quad (6)$$

if $\ell_1 \neq \ell_2$, and

$$\int_{-1}^{+1} P_\ell(x) P_\ell(x) dx = \frac{2}{2\ell + 1} . \quad (7)$$

With these it is straightforward to show from Eq. (4) that

$$c_\ell = \frac{2\ell + 1}{2} \int_{-1}^{+1} f(x) P_\ell(x) dx . \quad (8)$$

For example, if $f(x) = e^x$ we have

$$c_2 = \frac{5}{2} \int_{-1}^{+1} e^x \frac{1}{2} (3x^2 - 1) dx = 0.357814349 . \quad (9)$$

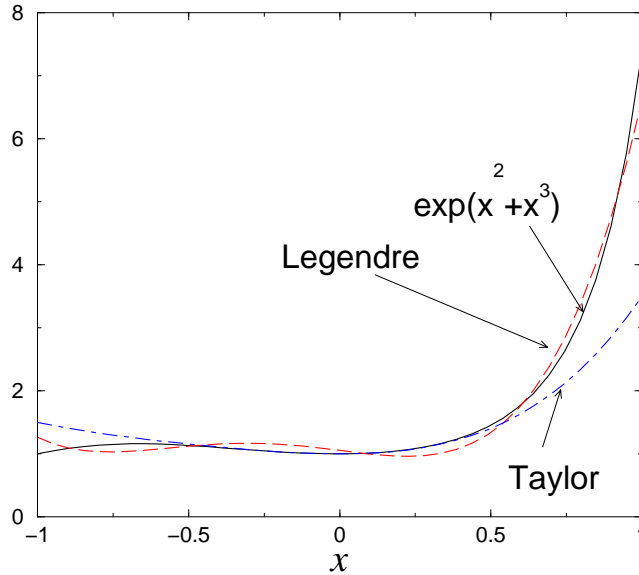


Figure 1: Approximation to the exponential function $e^{x^2+x^3}$, with two different fourth order polynomials. One is the Taylor series for the function; the other is the Legendre series for the function.

This is way the coefficient of $P_2(x)$ was found for Eq. (5).

PROBLEM 1 Show that Eq. (8) follows from Eq. (6) and Eq. (7).

PROBLEM 2 Use Maple to find the expansion of $f(x) = \sin(2\pi x)$ in terms of the first seven Legendre polynomials $P_0(x) \cdots P_6(x)$. Using Maple plot this approximation and $f(x)$ on the same graph, for the interval $-1 \leq x \leq +1$.

PROBLEM 3 Using the first 5 Legendre polynomials $P_0 \cdots P_4$. find an approximation for $\sin \theta$ on the interval $0 \leq \theta \leq \pi$.

PROBLEM 4 Use Maple to verify that $P_7(x)$ satisfies Eq. (2)

We are now ready for the most important remark about the use of a series of Legendre polynomials: If you use a series of Legendre polynomials as in Eq. (4), with the coefficients computed according to Eq. (8), then the more terms you use, the better the approximation will be, and if you use an infinite number of terms, the approximation will be perfect. (We won't dwell here on the subtleties

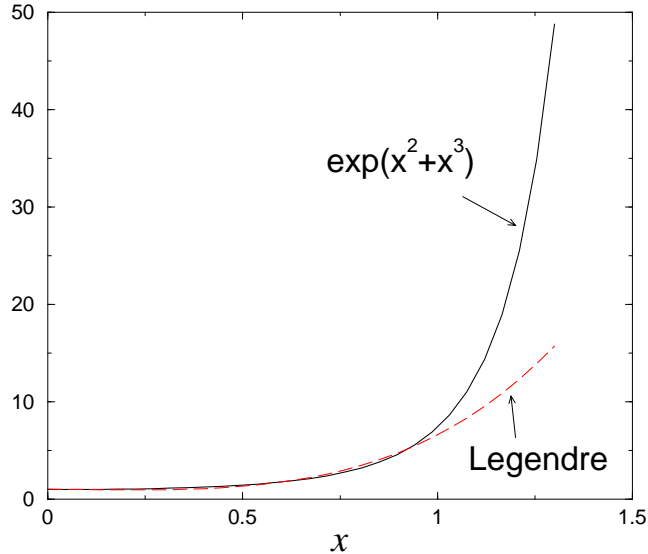


Figure 2: Accuracy of the Legendre approximation for $e^{x^2+x^3}$ outside the interval $[-1, +1]$.

of what “perfect” means. For our purposes, “perfect” will mean that the series gives the correct value of the function at every value of x .)

If you consider it for a moment, the process we have just described is strange and wonderful. We have taken a function $f(x)$ and replaced it by an infinite set of numbers c_0, c_1, c_2, \dots . If you know the function you can find the numbers (with Eq. (8)) ; if you know the numbers you can find the function (with Eq. (4)). This ability to switch back and forth will prove to be very useful.

FUNCTIONS AS VECTORS

You should be familiar with the way of writing a 3-dimensional vector in terms of its Cartesian components:

$$\mathbf{V} = V^x \hat{\mathbf{x}} + V^y \hat{\mathbf{y}} + V^z \hat{\mathbf{z}} , \quad (10)$$

although you might have seen it with the following notation:

$$\mathbf{V} = V^x \hat{\mathbf{i}} + V^y \hat{\mathbf{j}} + V^z \hat{\mathbf{k}} .$$

These equations state the fact that any 3-dimensional vector can be written as a sum of 3 “basis vectors.” The coefficients V^x, V^y, V^z in the sum are called the components, and as an immediate consequence of the fact that the basis vectors are orthonormal (orthogonal to each other and of unit length), the components can be found from

$$V^x = \hat{\mathbf{x}} \cdot \mathbf{V} \quad V^y = \hat{\mathbf{y}} \cdot \mathbf{V} \quad V^z = \hat{\mathbf{z}} \cdot \mathbf{V} . \quad (11)$$

It is useful, not only in electrodynamics, but in other areas of physics, to see a connection between the mathematics of vectors in Eqs. (10,11) and the representation of a function with Legendre functions in Eq. (4). The first part is obvious. In Eq. (4) we can view the “basis functions” to be P_0, P_1, P_2, \dots , and the “components of $f(x)$ ” to be c_0, c_1, c_2, \dots . There is, of course, an interesting new feature: We have an infinite number of basis functions P_0, P_1, P_2, \dots , so the dimension of the space is infinite.

To continue, to make a connection between “function vectors” and 3 - dimensional vectors, we need to have something to act as a dot product of function vectors. We do this in the following way: For two functions $f(x)$ and $g(x)$, we define the dot product between them to be

$$f \cdot g \equiv \int_{-1}^{+1} f(x)g(x) dx . \quad (12)$$

This is the dot product that we use in the context of Legendre polynomials, where we are interested in the interval $[-1,+1]$. The nature of the dot product in Eq. (12) can be generalized to other intervals, and in other ways.

With this definition of the dot product, the properties of the Legendre functions in Eq. (6) and Eq. (7) can be expressed

$$P_{\ell_1} \cdot P_{\ell_2} = 0 \text{ if } \ell_1 \neq \ell_2 \quad P_\ell \cdot P_\ell = 2/(2\ell + 1) , \quad (13)$$

so that the Legendre polynomials can be considered an (infinite dimensional!) basis that is orthogonal, but not orthonormal. (We could easily define a new set of Legendre polynomials with “correction” factors, so that they are orthonormal, but the standard Legendre polynomials are used widely; we will stick with them.) With our integral-based dot product, the formulas for the components of the function vector $f(x)$, equivalent to Eq. (8), is

$$c_\ell = \frac{2\ell + 1}{2} P_\ell \cdot f , \quad (14)$$

It is true that what we have done here with Eqs. (12, 13,14) is just a way of viewing things differently, of viewing functions as vectors, but in fact this is a very useful way of viewing them.

PROBLEM 5 The function $h(x)$ is defined to be $h(x) = x+x^2$. Find two functions that are orthogonal to $h(x)$ with respect to the dot product defined in Eq. (12).

PROBLEM 6 Consider the set of all polynomials of order 3 or less. From the following considerations we know that this is a vector space: If you add any two such “vectors” you get another “vector;” if you multiply any such “vector” by a number you get another vector. Clearly this is a 4 dimensional vector space, and the obvious basis for this space is $1, x, x^2, x^3$. Let the dot product for this space be the one given in Eq. (12). (a) Show that this is not an orthogonal basis for the 4 dimensional space. (b) Find an orthonormal basis for this 4 dimensional vector space. (Hint: Start with the first 4 Legendre polynomials and readjust their magnitude so that they have “unit length.”)

MULTIPOLES AND THE LAPLACIAN

Consider some azimuthally symmetric function $F(r, \theta)$ of spherical coordinates r and θ . For any particular value of r — call it r_{special} — the function is a function only of the variable $x = \cos \theta$ for $-1 \leq x \leq +1$, so we can write

$$\begin{aligned} F(r_{\text{special}}, \theta) &= a_0 P_0(\cos \theta) + a_1 P_1(\cos \theta) + a_2 P_2(\cos \theta) + \dots \\ &\equiv \sum_{\ell=0}^{\infty} a_{\ell} P_{\ell}(\cos \theta) . \end{aligned} \quad (15)$$

Of course, it doesn't matter what the value of r_{special} is. This procedure will work for any r_{special} . But if we change from one value of r_{special} — call it r_{spec1} — to another value r_{spec2} , the dependence on θ will change. That is, the functions $F(r_{\text{spec1}}, \theta)$ and $F(r_{\text{spec2}}, \theta)$, are different functions of θ . This means that the coefficients a_{ℓ} in a multipole representation like Eq. (15) will be different. In other words, the values of the a_{ℓ} will depend on the value of r_{special} . We could denote that by writing the dependence as $a_{\ell}(r_{\text{special}})$.

Now that we have established this dependence of the coefficients on the radial variable, we can drop the “special” notation, and write the general multipole expansion of a function $F(r, \theta)$ as

$$F(r, \theta) \equiv \sum_{\ell=0}^{\infty} a_{\ell}(r) P_{\ell}(\cos \theta) \equiv \sum_{\ell=0}^{\infty} a_{\ell}(r) P_{\ell}(x) . \quad (16)$$

In the last equality we have reintroduced the useful notation $x = \cos \theta$. (Of course, this “ x ” has nothing to do with the $x = r \sin \theta \cos \phi$ of Cartesian coordinates; it is just notation.)

This multipole decomposition of a function is especially useful if we will be looking at the Laplacian of the function $F(r, \theta)$. In spherical coordinates the Laplacian operator has the form

$$\nabla^2 F = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dF}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{dF}{d\theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{d^2 F}{d\phi^2} . \quad (17)$$

We are concerned for now only with functions independent of ϕ , so we will drop the last term. We will also replace the θ derivatives with derivatives with respect to $\cos\theta$ and we will use the notation $x \equiv \cos\theta$. With these changes, Eq. (17) becomes

$$\nabla^2 F = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dF}{dr} \right) + \frac{1}{r^2} \frac{d}{dx} \left((1-x^2) \frac{dF}{dx} \right) \quad (18)$$

Since *any* function $F(r, \theta)$ can be written in the multipole form in Eq. (16), we now use that expression in Eq. (18):

$$\begin{aligned} \nabla^2 F &= \nabla^2 \left[\sum_{\ell=0}^{\infty} a_{\ell}(r) P_{\ell}(x) \right] = \sum_{\ell=0}^{\infty} \nabla^2 [a_{\ell}(r) P_{\ell}(x)] \\ &= \sum_{\ell=0}^{\infty} \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{da_{\ell}(r)}{dr} \right] P_{\ell}(x) + \frac{a_{\ell}(r)}{r^2} \frac{d}{dx} \left[(1-x^2) \frac{dP_{\ell}(x)}{dx} \right] \\ &= \sum_{\ell=0}^{\infty} \left\{ \frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{da_{\ell}(r)}{dr} \right] - \ell(\ell+1) \frac{a_{\ell}(r)}{r^2} \right\} P_{\ell}(x) \end{aligned} \quad (19)$$

In dealing with static electric and magnetic fields (as well as *many* other physical fields) we are often interested in functions that are “harmonic,” i.e., functions which have a zero Laplacian. If the azimuthally symmetric field F in Eq. (19) is harmonic, then we must have

$$\frac{1}{r^2} \frac{d}{dr} \left[r^2 \frac{da_{\ell}(r)}{dr} \right] - \ell(\ell+1) \frac{a_{\ell}(r)}{r^2} = 0 \quad \text{for all } \ell .$$

It follows that the only possible solutions for $a_{\ell}(r)$ are

$$a_{\ell}(r) = \text{const} \times r^{\ell} \quad \text{or} \quad a_{\ell}(r) = \text{const} \times r^{-\ell-1} . \quad (20)$$

These constants can be different for different values of ℓ of course, so that the general (azimuthally symmetric) solution for $\nabla^2 F = 0$ is

$$F(r, \theta) \equiv \sum_{\ell=0}^{\infty} [\alpha_{\ell} r^{\ell} + \beta_{\ell} r^{-\ell-1}] P_{\ell}(\cos\theta) , \quad (21)$$

where the α_{ℓ} and β_{ℓ} are any constants.

PROBLEM 7 According to Eq. (21) the function $r^2 P_2(\cos\theta)$ must be harmonic. Find the form of this function in Cartesian coordinates and show explicitly, in Cartesians, that it satisfies Laplace’s equation.

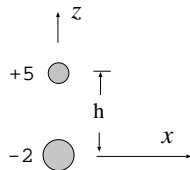


Figure 3: A simple charge distribution.

MULTIPOLE MOMENTS FOR THE ELECTROSTATIC POTENTIAL

Consider the very simple distribution of charges shown in Fig. 3. A point charge of -2 coulombs is at the origin, and a +5 coulomb charge is on the z axis at $z = h$. You should be able to write down the electrostatic potential produced by these charges with no problem:

$$\Phi(x, y, z) = \frac{1}{4\pi\epsilon_0} \left(-2 \frac{1}{\sqrt{x^2 + y^2 + z^2}} + 5 \frac{1}{\sqrt{x^2 + y^2 + (z - h)^2}} \right). \quad (22)$$

It is not hard to convert this expression to one using polar coordinates. (Remember that $x^2 + y^2 = r^2 \sin^2 \theta$ and $z = r \cos \theta$.) Since the potential is clearly azimuthally symmetric, it is clear that there will be no dependence on the azimuthal angle ϕ . The result is

$$\Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \left(-2 \frac{1}{r} + 5 \frac{1}{\sqrt{r^2 - 2h \cos \theta + h^2}} \right). \quad (23)$$

Suppose that we wanted an approximation that is accurate at large distances r . The obvious thing to do is to expand Eq. (22) in a Taylor series in $1/r$. The result (that you can easily confirm using Maple) is:

$$\begin{aligned} \Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} & \left(\frac{3}{r} + \frac{5h}{r^2} \cos \theta + \frac{5h^2}{r^3} \left[\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right] \right. \\ & \left. + \frac{5h^3}{r^4} \left[\frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right] + \dots \right). \end{aligned} \quad (24)$$

Notice that this can be rewritten as

$$\begin{aligned} \Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} & \left(\frac{3}{r} P_0(\cos \theta) + \frac{5h}{r^2} P_1(\cos \theta) + \frac{5h^2}{r^3} P_2(\cos \theta) \right. \\ & \left. + \frac{5h^3}{r^4} P_3(\cos \theta) + \dots \right). \end{aligned} \quad (25)$$

If we kept more terms, we would find that the obvious pattern continues.

It should not come as a surprise that the potential can be written in such a form. Since the potential Φ must satisfy Laplace's equation (where there is no charge), then it must have the form given in Eq. (21). Intuition tells us that the potential must decrease with increasing distance r from the source of the fields, so all the α s must be zero. (The single exception is α_0 , but the corresponding term $\alpha_0 r^0 P_0(\cos \theta) = \alpha_0$ in Eq. (21) is just a constant. Setting this constant to zero is equivalent to choosing the potential to be zero at $r = \infty$, the usual choice.) In view of this the potential *must* have the form given in Eq. (24). Except for the values of the constant multipliers, it was a foregone conclusion!

Let us write the potential (in the charge free region outside a distribution of charge) as

$$F(r, \theta) \equiv \sum_{\ell=0}^{\infty} \frac{\beta_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta) . \quad (26)$$

This is much more than a mathematical curiosity; it is a very useful way to look at fields, especially at places far from the sources of the fields. Think of the field at a large distance from the charge distribution, so that we are considering $F(r, \theta)$ for large r . The expression in Eq. (26) tells us that for sufficiently large r the β_0/r term will be much larger than any of the other terms. We know, therefore, that at sufficiently large r the field will be spherically symmetric (θ independent) and that we only need a single number, the value of β_0 , in order to have a good approximation for the field. (If we want a somewhat better approximation, or if we want to know approximately the variation in the field at different values of θ we could also compute β_1 .) This simple conclusion — you only need to know β_0 — no longer applies, of course, if $\beta_0 = 0$. In this case we conclude that the field falls off with distance as $1/r^2$, that the angular dependence of the field is $\Phi \propto \cos \theta$, and that we need to know β_1 to have an approximation of the field at large r . This pattern continues to higher multipoles. If $\beta_1 = 0$, we need to know β_2 and so forth.

We now turn to the question of how to determine the coefficients. The general expression for determining the potential at any point in space — denoted \vec{r} — from a known distribution of charge, is with the integral

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} d\text{Volume}' \quad (27)$$

where ρ is the charge per unit volume. In the case of point charges q_k at locations \vec{r}_k , we replace Eq. (27) by

$$\Phi(\vec{r}) = \frac{1}{4\pi\epsilon_0} \sum_k \frac{q_k}{|\vec{r} - \vec{r}_k|} . \quad (28)$$

There is a straightforward way to evaluate the β coefficients from these expressions. We shall call this “method 1.” In order not to delay the application of

multipole methods to physical problems we shall not give a mathematical justification for method 1. This would require some detailed manipulations, but not any unacceptably advanced mathematics. In method 1, each coefficient in Eq. (26) is given by

$$\beta_\ell = \frac{1}{4\pi\epsilon_0} \int \rho(r, \theta) P_\ell(r, \theta) r^\ell d\text{Volume} . \quad (29)$$

The first few β s have simple interpretations. Note that according to Eq. (29) β_0 is given by

$$\beta_0 = \frac{1}{4\pi\epsilon_0} \int \rho(r, \theta) d\text{Volume} = \frac{1}{4\pi\epsilon_0} (\text{total charge}) . \quad (30)$$

We say that the total charge is the monopole (or $\ell = 0$) multipole of the charge distribution.

According to Eq. (29) The coefficient β_1 is given by

$$\beta_1 = \frac{1}{4\pi\epsilon_0} \int \rho(r, \theta) r \cos \theta d\text{Volume} = \frac{1}{4\pi\epsilon_0} \int z \rho(r, \theta) d\text{Volume} . \quad (31)$$

This equation tells us that in the integral each bit of charge is weighted by its z position. In the case that the charge distribution consists of a set of discrete charges q_k located along the z axis (so that the distribution is azimuthally symmetric), the integral turns into

$$\beta_1 = \frac{1}{4\pi\epsilon_0} \sum_k q_k z_k . \quad (32)$$

The integral in Eq. (31) or the sum in Eq. (32) is called the “dipole moment” of the source and is denoted “ p .” In the case of the configuration in Fig. 3, the sum in Eq. (32) gives us $p = (-2) \times (0) + (+5) \times h = 5h$, and the dipole term in the potential is precisely as it appears in Eq. (25). Notice that this term can be written as

$$\Phi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0} \frac{5h}{r^2} P_1(\cos \theta) = \frac{1}{4\pi\epsilon_0} \frac{p \cos \theta}{r^2} = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} . \quad (33)$$

In this last expression \vec{p} called the “dipole vector” is $p \hat{z}$ where \hat{z} is the unit vector in the z direction, and \hat{r} is a unit vector pointing to the position r, θ at which the value of $\Phi(r, \theta)$ is being specified. The advantage of this vectorial notation for the dipole is that it turns out to apply also to cases with non-azimuthally symmetric distributions of charge. In that case the vector \vec{p} will not necessarily point in the z direction.

For $\ell = 2$, the expressions for β are

$$\beta_2 = \frac{1}{4\pi\epsilon_0} \int \rho(r, \theta) r^2 \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) d\text{Volume}$$

$$= \frac{1}{4\pi\epsilon_0} \int \rho(r, \theta) \left(\frac{3}{2}z^2 - \frac{1}{2} \right) d\text{Volume} , \quad (34)$$

or

$$\beta_2 = \frac{1}{4\pi\epsilon_0} \sum_k q_k \left(\frac{3}{2}z_k^2 - \frac{1}{2} \right) . \quad (35)$$

The integral in Eq. (34) or the sum in Eq. (35) is called the quadrupole moment of the source (aside from factors of 2 that are different in different conventions).

A second method for finding the β s is a trick that is often very efficient. It is best introduced with an example. For the charge distribution in Fig. 3, the potential on the z axis, for $z > h$ is

$$\Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \left(-2\frac{1}{z} + 5\frac{1}{z-h} \right) = \frac{1}{4\pi\epsilon_0} \left(-2\frac{1}{z} + 5\frac{1}{z} \frac{1}{1-h/z} \right) . \quad (36)$$

We now use the “geometric series.” For a number δ the following expression is valid provided only that $|\delta| < 1$

$$\frac{1}{1-\delta} = 1 + \delta + \delta^2 + \delta^3 + \dots = \sum_{k=0}^{\infty} \delta^k . \quad (37)$$

Since $h/z < 1$ we can use this series for the last term in Eq. (36) and write the result as

$$\Phi(r, \theta) = \frac{1}{4\pi\epsilon_0} \left[-2\frac{1}{z} + 5\frac{1}{z} \sum_{k=0}^{\infty} \left(\frac{h}{z} \right)^k \right] . \quad (38)$$

Now the positive z axis corresponds to $\cos\theta = 0$ and $r = z$, so that the general expression in Eq. (26) becomes

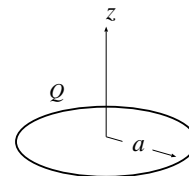
$$F(r, \theta) \equiv \sum_{\ell=0}^{\infty} \frac{\beta_{\ell}}{z^{\ell+1}} , \quad (39)$$

where we have used the fact that $P_{\ell}(\cos 0) = P_{\ell}(1) = 1$. From a comparison of Eq. (39) and Eq. (38) we conclude that

$$\beta_0 = \frac{3}{4\pi\epsilon_0} \quad \text{and} \quad \beta_{\ell} = \frac{5h^{\ell}}{4\pi\epsilon_0} \quad \text{for } \ell > 0 . \quad (40)$$

We end with a comment on the role played by the location of the origin in the determination of the multipole moments. As an example of what this means, consider the charge distribution in Fig. 3, but with the origin translated upward so that the +5 coulomb charge is at $z = 0$ and the -2 coulomb charge is at $z = -h$. It is straightforward to calculate anew the β s for this “new” charge distribution and to find $\beta_0 = +3$, $\beta_1 = -2h$, $\beta_2 = -2h^2 \dots$. Since we can change the values of the β s simply by changing the choice of origin, and without

making any physical change, it may seem that there is no physical meaning to the multipole moments. But consider this: we *cannot* change the value of β_0 by moving the origin, and it is this value that carries the important information about the field far from the source. To get a deeper understanding of what is going on here, change the -2 coulomb charge in Fig. 3 to a charge of -5 coulomb, so that (as you should easily be able to verify) $\beta_0 = 0$ and β_1 is unchanged. Now if we move the origin neither β_0 nor β_1 changes. Now, also, it is β_1 that is important in determining the strength of the distant field. This general pattern extends to larger multipoles. If, for example, $\beta_0 = 0$, $\beta_1 = 0$, $\beta_2 = 0$ and $\beta_3 = 0$ then the value computed for β_4 will be independent of where the origin is placed. These issues will be explored further in the problems.



PROBLEM 8 A ring of radius a has an electrical charge Q uniformly distributed over it. Find the monopole, dipole and quadrupole moments for this charge distribution (equivalently, find $\beta_0, \beta_1, \beta_2$ by “method 1” described in the text).

PROBLEM 9 For the ring described in the previous problem find the electrical potential on the z axis by using standard techniques (adding bits of charge divided by the distance to the charge). Using this expression on the z axis, find the values of $\beta_0, \beta_1, \beta_2$ and compare with the values found in the previous problem.

PROBLEM 10 For the uniformly charged ring described above, use Maple to find β_{16} .

PROBLEM 11 Suppose a point charge with +1 coulomb is at $z = 1$ m; a charge of -1/4 coulomb is at $z = 2$ m; a charge of +1/9 coulomb is at $z = 3$ m; a charge of -1/16 coulomb is at $z = 4$ m; and so forth. Find the dipole moment of this distribution.

PROBLEM 12 Suppose that a certain (azimuthally symmetric) charge distribution has a total charge of 7 coulombs and is confined to a region of a few centimeters. Suppose that the potential at $z = 5$ m is larger than the potential at $z = -5$ m by 4%. Find the dipole moment of the charge distribution.

PROBLEM 13 Suppose that the ring of uniform charge Q and radius a is placed at $z = -3a$. Find the value of β_1 . (Hint: Do *not* do it by evaluating an integral.)

PROBLEM 14 Suppose that the monopole moment (i.e., the total charge) of an azimuthally symmetric distribution is nonzero. (a) Show that the origin can always be shifted to make the dipole moment zero. (b) Show that the origin can be shifted to make β_2 zero. (c) Show that in general the origin cannot be shifted to make both β_1 and β_2 zero for the same location of the origin.

THE MAGNETIC SCALAR POTENTIAL

The electrostatic field \vec{E} satisfies two basic equations

$$\nabla \times \vec{E} = 0 \quad \nabla \cdot \vec{E} = \rho/\epsilon_0 \quad (41)$$

The first of these equations is equivalent to the statement that the electrostatic field is conservative — that its line integral around a closed loop gives zero. This allows us to define the electrostatic potential Φ by the equation

$$\vec{E} = -\nabla\Phi . \quad (42)$$

The second equation in Eq. (41) then ends up telling us the value of $\nabla^2\Phi$.

The magnetostatic field \vec{B} is not conservative. Its curl is not zero, but rather

$$\nabla \times \vec{B} = \mu_0\vec{J} , \quad (43)$$

where \vec{J} is the current density. The line integral of \vec{B} around a closed curve is not zero. Rather we have Ampère's Law: for a loop, the line integral of \vec{B} around that loop is given by

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I , \quad (44)$$

where I is the current through the loop. We cannot therefore define a potential

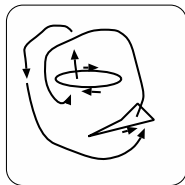


Figure 4: A region of confined currents.

for \vec{B} in the same worry-free way that we did for \vec{E} . For this reason the kind of potential usually used for magnetostatics is the magnetic *vector potential*. But the advantages of working with a scalar potential are so great, that we *will* introduce a scalar potential, although it cannot handle all situations. Suppose we have a confined region of currents, as shown in Fig. 4, and suppose we scrupulously avoid the interior of that region, remaining in the current-free region outside. In our current free region the curl of \vec{B} will be zero, and the line integral of \vec{B} around any loop will be zero. Of course the loop must remain outside the region of confined currents. By avoiding the current-containing

region in this way we have a sort of limited conservative property for \vec{B} , and we can define a “magnetic scalar potential” Φ_m , in analogy with the electrostatic potential, by

$$\vec{B} \equiv -\nabla\Phi_m . \quad (45)$$

Of course, there is no free lunch. By using Φ_m we get a convenient description of the magnetic field outside the region of currents, but — since Φ_m has no meaning inside the region of currents — we cannot directly relate Φ_m to the sources of the magnetic field. In other words, there is no convenient general rule, like that in Eq. (27) or Eq. (28). In order to find Φ_m we must find the magnetic field some other way (e.g., by the Biot-Savart law, or with the magnetic vector potential, or ...).

Since the divergence of \vec{B} is zero, it follows immediately from Eq. (45) that wherever Φ_m exists it satisfies

$$\nabla^2\Phi_m = 0. \quad (46)$$

As we did for electrostatics, let us suppose that the sources for the magnetic field are azimuthally symmetric, and therefore that the field is azimuthally symmetric. Since Φ_m is a harmonic function we can immediately conclude that (where it exists, outside a current region) Φ_m can be written in the same form as Eq. (26)

$$\Phi_m(r, \theta) \equiv \sum_{\ell=0}^{\infty} \frac{\gamma_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta) . \quad (47)$$

(We use the notation γ for the coefficients to help avoid confusion with the coefficients β in the electrostatic potential.) In other words, the magnetostatic field can be described in multipoles in complete analogy with electrostatic multipoles.

In electrostatics we gave two methods for finding the multipole moments, the coefficients of a multipole expansion. One of the methods, “method 1,” was based on the integral in Eq. (27) or the sum in Eq. (28). There are no corresponding relations for Φ_m , so we must usually rely in general on “method 2,” finding the field on the z axis by other means.

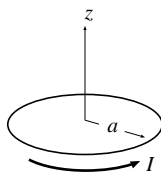


Figure 5: A ring of current in the xy plane.

As an important example we now work out the multipoles for a ring of radius a carrying a current I , as illustrated in Fig. 5. (We will later compare this with the multipoles of a ring of charge, as worked out in the problems.) We place the ring in the xy plane centered on the origin. On the z axis, by symmetry, the magnetic field points in the $+z$ direction. It is straightforward, and a very standard problem in introductory courses, to work out the details of the field on the z , and the result for the z component of \vec{B} is

$$B^z = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}}. \quad (48)$$

We can now make an expansion of this expression for $z > a$ and we get

$$\begin{aligned} B^z &= \frac{\mu_0 I a^2}{2z^3} \left[1 + \left(\frac{a}{z}\right)^2 \right]^{-3/2} \\ &= \frac{\mu_0 I a^2}{2z^3} - \frac{3\mu_0 I a^4}{4z^5} + \frac{15\mu_0 I a^6}{16z^7} + \dots \end{aligned} \quad (49)$$

We want to compare this to a multipole expansion, so let us write the general expression for the azimuthally symmetric multipole expansion, Eq. (47), but let us confine it to the positive z axis, so that $r = z$ and $P_\ell(\cos\theta) = P_\ell(1) = 1$. (Recall that this is a defining property of the Legendre polynomials.) In this case the multipole expansion reduces to

$$\Phi_m(z, 0) \equiv \sum_{\ell=0}^{\infty} \frac{\gamma_\ell}{z^\ell} = \frac{\gamma_0}{z} + \frac{\gamma_1}{z^2} + \frac{\gamma_2}{z^3} + \dots, \quad (50)$$

and hence, on the z axis

$$B^z = -\frac{\partial\Phi_m}{\partial z} = \frac{\gamma_0}{z^2} + 2\frac{\gamma_1}{z^3} + 3\frac{\gamma_2}{z^4} + \dots \quad (51)$$

By comparing this expression with Eq. (49) we can read off the values of the multipole moments. We see that all the γ_ℓ with odd index are zero, and that

$$\gamma_1 = \frac{\mu_0 I a^2}{4} \quad \gamma_3 = -\frac{3\mu_0 I a^4}{16} \quad \gamma_5 = \frac{5\mu_0 I a^6}{32}, \quad (52)$$

and so forth.

As an example of the usefulness of this result, consider that at large distances you really only need to know the dipole moment; it is much larger than the other multipole moments. [For example, from Eq. (49) you can see that at a distance of $z = 4a$ the octupole magnetic field on the z axis is only 10% as large as the dipole part.] In general, the dipole part of the magnetic scalar potential is written as

$$\Phi_{\text{dip}} = \frac{\mu_0}{4\pi} \frac{m \cos\theta}{r^2}. \quad (53)$$

From our results above we can infer that

$$m = \pi a^2 I , \quad (54)$$

and that the dipole part of the magnetic scalar potential for the circular loop is

$$\Phi_{\text{dip}} = \frac{\mu_0 I a^2 \cos \theta}{4 r^2} . \quad (55)$$

By comparison

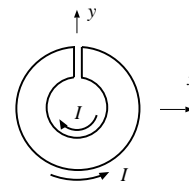
$$\Phi_{\text{dip}} = \frac{\mu_0 I a^2 \cos \theta}{4 r^2} = \frac{\mu_0 I a^2}{4} \frac{z}{(x^2 + y^2 + z^2)^{3/2}} . \quad (56)$$

From this we can compute all the components, B^x , B^y and B^z .

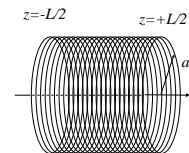
This example shows how the multipole formalism lets you start with only the z component of \vec{B} on the z axis, and find all components of all multipoles everywhere.

To end this section we give, without proof, a very useful result. For planar loops of current carrying wire, the magnitude of the dipole moment produced is just the area enclosed by the loop, multiplied by the current in the wire. If there are N turns of wire, it is the same as if the current were NI . This rule works for planar circuits that are not axisymmetric. If the current source consists of a more complicated array of wire, then superposition can often be used to reduce the problem to one of superposing several loops. It is even possible to use an extension of this rule for circuits that are not planar.

PROBLEM 15 A circuit consists of a loop, in the xy plane, of radius R , with N turns of wire carrying current I in the clockwise direction connected to a loop of radius $2R$ carrying the current in the counterclockwise direction. Find the magnetic dipole moment for this configuration.



PROBLEM 16 The solenoid shown in the figure is of length L and radius a . It is centered on the coordinate origin and carries a current I through N turns of wire. For $z > L/2$ the magnetic field can be written as an expression like that in Eq. (51). Find the values of the γ_k up to $k = 5$.



HELMHOLTZ COILS

In certain applications, especially in magnetic resonance imaging (MRI), it is useful to have the magnetic field be as nearly uniform as possible. That

means that the direction and the magnitude should be — insofar as possible — independent of position in some region. The usual way of doing this is to use two symmetric coils (carrying equal current in the same sense, with equal numbers of turns, in parallel planes, as shown in Fig. 6. In that figure the coils, have radius a , with N turns of wire carrying current I . They are separated by a distance L , and are located symmetrically about the origin at $z = \pm L/2$. The coils are assumed to have negligible length compared to their radius, as if all the turns were bunched together. The region of interest, the region in which we want the magnetic field to be nearly uniform, is the region around the origin. If we compute the field on the z axis, by superposing two expressions like that

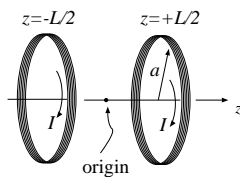


Figure 6: Symmetrically arranged coils.

in Eq. (48), we get:

$$B^z = \frac{N\mu_0 I a^2}{2[a^2 + (z - L/2)^2]^{3/2}} + \frac{N\mu_0 I a^2}{2[a^2 + (z + L/2)^2]^{3/2}}. \quad (57)$$

Since we are interested in the region near the $z = 0$ origin, it is useful to make a Taylor series for small z . In general such a series would have the form

$$B^z = C_0 + C_1 z + C_2 z^2 + C_3 z^3 + C_4 z^4 + \dots \quad (58)$$

Due to the symmetry of the arrangement illustrated in Fig. 6, only the terms that are even powers of z are nonzero, and we are left with

$$B^z = C_0 + C_2 z^2 + C_4 z^4 + C_6 z^6 + \dots \quad (59)$$

This already shows that the field is “slightly” uniform. A truly uniform field would have $B^z = C_0$ everywhere. The expression in Eq. (59) shows that the actual expression is not just $B^z = C_0$, but has small corrections for small values of z . Because the corrections are quadratic in z they are, in fact, *very* small corrections for small z . Another way of understanding this “slightly” uniform nature of the magnetic field is to notice that the z derivative of B^z in Eq. (59) is zero at $z = 0$.

It is possible to make the magnetic field considerably more uniform. To see how to do this, we need to look at the actual values of the coefficients in

Eq. (59):

$$\begin{aligned}
 C_0 &= 8 \frac{N\mu_0 I a^2}{(4a^2 + L^2)^{3/2}} \\
 C_2 &= 192 \frac{N\mu_0 I a^2 (L^2 - a^2)}{(4a^2 + L^2)^{7/2}} \\
 C_4 &= 1920 \frac{N\mu_0 I a^2 (2a^4 - 6a^2L^2 + L^4)}{(4a^2 + L^2)^{11/2}} . \tag{60}
 \end{aligned}$$

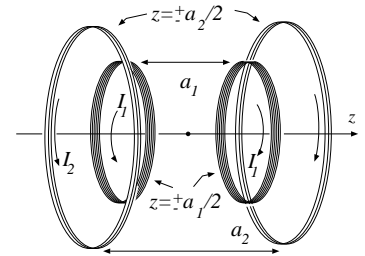
The important thing to notice is that we can adjust the geometry so that C_2 is zero. If we make $L = a$, that is, if we make the separation equal to the radius, then C_2 is zero and the magnetic field near the origin is very nearly constant. When a pair of coils is arranged in this way, with separation of the planes equal to the radius of the coils, they are called *Helmholtz coils*.

PROBLEM 17 The expansion in Eq. (58) applies to small values of z (more specifically, to $z \ll a$). For the coils (radius a , separation L) shown in Fig. 6, find an expansion of the form

$$B^z = D_0 + D_1/z + D_2/z^2 + D_3/z^3 + D_4/z^4 + \dots . \tag{61}$$

(a) Find the values of D_k for k from 0 to 8. (b) In the special case of Helmholtz coils ($L = a$) show that there is no octupole moment, that is, show that the value of D_5 turns out to be zero.

PROBLEM 18 When Helmholtz coils are used to create a uniform strong magnetic field for magnetic resonance imaging (“MRI”), it is important that the magnetic field at a moderate distance from the imaging hardware be weak. To accomplish this, *dual* Helmholtz coils are used. This is shown in the figure to the right. The inner Helmholtz coil has radius (and separation) a_1 and N_1 turns of wire carrying current I_1 ; the Helmholtz coil has radius (and separation) a_2 and N_2 turns of wire carrying current I_2 . If the parameters of the configuration are adjusted so that there is no dipole moment, then the lowest nonzero multipole is the $\ell = 5$ multipole.



Design a practical dual Helmholtz coil apparatus that produces a field of 100 gauss at its center. (Give the number of turns, the current and the size of both pairs of coils.) For your design, find the strength of the magnetic field, in the xy plane (the plane perpendicular to the axis), at a distance from the origin equal to three times the radius of the larger coil.

PROBLEM 19 For the dual Helmholtz coils you designed in the previous problem, use Maple to plot the on-axis z component of the magnetic field as a function of z in two ways: First, using the exact expression based on the Biot-Savart law, and second, by assuming that the magnetic field is a purely $\ell = 5$ field. Give the plot only for values of z larger than the location of the plane of the outer coil. At what value of z is the “pure $\ell = 5$ approximation” accurate to within 1% of the value of the field at the origin.

PROBLEM 20 Suppose that a solenoid of radius a and length $2a$ has an even number of turns N of wire. The solenoid can be considered to be a superposition of $N/2$ Helmholtz coils, each with one turn of wire as shown in the figure. The octupole moment of a Helmholtz coil is zero (as you showed in Prob. 18), so the solenoid — since it’s a superposition of Helmholtz coils — should have no octupole moment. But in Prob. 17 you found the value of ξ , and it was not zero. Explain.

LABORATORY/MODELLING PROBLEM You are given a set of dual Helmholtz coils and an uncalibrated magnetometer that can be used to measure magnetic field strength. The coils have the number of turns marked on them. You will be shown how to measure the current going into either of the coils by measuring the voltage drop across an accurate resistor that is in series with the coils.

(a) From the model for Helmholtz coils, compute the theoretical magnetic field strength at the center of the Helmholtz coil, in terms of the number of turns, the radius, and the current. Use this model and the smaller pair of Helmholtz coils to calibrate your magnetometer. This calibration should take the form “number of gauss in the B field, per volt of magnetometer reading.”

(b) Put some current in the large Helmholtz pair of coils, and use your (now calibrated) magnetometer to measure the field at the center. Compare this with your theoretical model of what the field should be.

(c) Write a Maple program that computes the magnetic field along the axis of a Helmholtz coil. Have the program plot the magnetic field (in units of the central magnetic field) as a function of “ z ,” the distance from the center, along the axis, in units of the coil radius. On the plot, include graphing lines, so that you can use the printed output as graph paper to compare the theoretical prediction with the measured value. You will be given a Maple program that illustrates how to add graph lines.

(d) Measure the field on the axis of the small Helmholtz pair as a function of z . You should remove the magnetometer from its track to get some points at large distances. Plot your measurements on the “graph paper” from part (c). [Hint: You can defer these measurement and make them at the same time as you make the measurements in part (f).]

(e) Arrange the two Helmholtz pairs to be concentric, and place the magnetometer on axis, at a large distance from the center. Hold the current to the small coil constant and adjust the current to the large coil until the magnetometer measures a minimum field. Compare the current ratio for this minimum to the theoretical current ratio that gives a minimum on-axis field at very large distances.

(f) With the large pair of coils shielding the small pair (i.e., with the current adjusted for a minimum at the largest distances) measure the on-axis field as a function of z .

(g) Write a Maple program [analogous to that in part (c)] to compute the on-axis field of the pair of Helmholtz coils with current ratio adjusted so that the net dipole moment of the two pairs of coils is zero. Have your program plot the logarithm (base 10) of the on-axis field as a function of z . The magnetic field should be normalized by the value of the unshielded central field, and z should be normalized by the radius of the small coil. Have your Maple program add grid lines to the output plot, and use the output as graph paper on which you plot the data from part (f).

- (h) With current supplied only to the inner coil, measure and record the magnetic field (normalized by the central field) in the “equatorial plane” of the coil, at distances from the axis ranging from 25 cm to at least 70 cm.
- (i) Using the idea of multipoles, compute the theoretical magnetic field strength in the equatorial plane, and use Maple to plot the field strength, normalized by the theoretical central field. On the plot include coordinate grid lines. On the resulting “graph paper” mark in your measurements from part (h), and compare theory and measurement.
- (j) Repeat part (h) in the case of “shielding” (i.e., in the case that the outer coil is made concentric with the inner coil and its current is adjusted to null the dipole moment of the pair of Helmholtz coils.)
- (k) Repeat part (i) in the case of “shielding.” The normalization here (the central field value by which you divide) can be taken to be the unshielded value of the central field.